Galois theory, Problems 4

To be handed in by 1:00 pm, 22 November 2017

1. Suppose that $E : K$ and $F : K$ are finite extensions having the property that K, E and F are contained in a field L.

   (i) Show that $EF : K$ is a finite extension;

   (ii) Show that when $E : K$ and $F : K$ are both normal, then $E \cap F : K$ is a normal extension;

   (iii) Show that when $E : K$ and $F : K$ are both normal, then $EF : E \cap F$ is a normal extension.

2. Which of the following field extensions are normal?

   (i) $\mathbb{Q}(\sqrt{3}) : \mathbb{Q}$

   (ii) $\mathbb{Q}(\sqrt[3]{3}) : \mathbb{Q}$

   (iii) $\mathbb{Q}(\sqrt{-1}) : \mathbb{Q}$

   (iv) $\mathbb{Q}(\sqrt{3}, \sqrt[3]{3}) : \mathbb{Q}$

   (v) $\mathbb{Q}(\sqrt{-1}, \sqrt{3}, \sqrt[3]{3}) : \mathbb{Q}$.

   Justify your answers.

3. Suppose that $L : M$ is an algebraic extension with $M \subseteq L$. Show that when $\alpha \in L$ and $\sigma : M \to \overline{M}$ is a homomorphism, then $\sigma(m_\alpha(M))$ is separable over $\sigma(M)$ if and only if $m_\alpha(M)$ is separable over M.

4. (a) Suppose that $f \in K[t]$ is separable over K and that $L : K$ is a splitting field extension for f. Show that $L : K$ is separable.

   (b) Suppose that $L : K$ is a splitting field extension for $S \subseteq K[t]$ where each $f \in S$ is separable over K. Show that $L : K$ is a separable extension.

5. Let p be a prime number, let $\mathbb{F}_p$ denote the finite field of p elements, and let $K = \mathbb{F}_p(t)$. Suppose that $L : K$ is a field extension, and $s \in L$ is transcendental over K.

   (a) Write $J = K(s)$, and let E denote a splitting field for the polynomial $x^p - t \in J[x]$. Show that for some $\xi \in E$, one has $x^p - t = (x - \xi)^p$, and deduce that $[E : J] = p$.

   (b) Let $U : J$ be a splitting field extension for the polynomial $(x^p - t)(x^p - s)$. By considering a splitting field extension F for the polynomial $x^p - s \in E[x]$, show that $[U : J] = p^2$.

6. With the same notation as in the previous question:

   (a) Show that if $\gamma \in U$, then $\gamma^p \in J$.

   (b) What is the degree of the field extension $J(\gamma) : J$? Explain.

   (c) Deduce that $U : J$ is a finite field extension which is not simple.