Galois theory, Problems 5

To be handed in 6th December 2017

1. Suppose that $L : M : K$ is an algebraic tower of fields. Prove that $L : K$ is separable if and only if $L : M$ and $M : K$ are both separable.

2. Suppose that $E : K$ and $F : K$ are finite extensions with $E \subseteq L$ and $F \subseteq L$, with L a field.

   (a) Show that when $E : K$ is separable, then so too is $EF : F$.

   (b) Show that when $E : K$ and $F : K$ are both separable, then so too are $EF : K$ and $E \cap F : K$.

3. Let f denote the polynomial $t^3 - 7$.

   (i) Write down a splitting field extension for f over $\mathbb{Q}$.

   (ii) Show that $\mathrm{Gal}_{\mathbb{Q}}(f) \cong S_3$.

   (iii) Use the Galois correspondence to determine all subfields of the splitting field that you wrote down in part (i). Draw the lattice of subfields and corresponding lattice of subgroups of S_3.

4. Let f denote the polynomial $t^3 + t + 1$.

   (i) Write down a splitting field extension for f over $\mathbb{F}_2$.

   (ii) What is $\mathrm{Gal}_{\mathbb{F}_2}(f)$? Justify your answer, and determine all subfields of the splitting field that you wrote down in part (i).

5. Let $L : K$ be a finite Galois extension with Galois group G. For any $\alpha \in L$, define the polynomial $f_\alpha(t) = \prod_{\sigma \in G}(t - \sigma(\alpha))$.

   (i) Show that $f_\alpha \in K[t]$.

   (ii) Prove that if $\sigma(\alpha) \neq \tau(\alpha)$ whenever $\sigma, \tau \in G$ satisfy $\sigma \neq \tau$, then $f_\alpha = m_\alpha(K)$.

   (iii) Use part (ii) to calculate the minimal polynomial of $2\sqrt{-3} - \sqrt{2}$ over $\mathbb{Q}$.

6. Suppose that L is a finite field having p^n elements, where p is a prime number. Recall that $\mathrm{Gal}(L : \mathbb{F}_p) = \langle \varphi \rangle$, where φ denotes the Frobenius mapping. Show that whenever K is a subfield of L, then $|K| = p^d$ for some divisor d of n. Show further that for each divisor d of n, there is a unique subfield K of L.