

UNIVERSITY OF BRISTOL

School of Mathematics

Examination for the Degree of B.Sc. and M.Sci. (Level M)

**GALOIS THEORY**

MATH M2700

(Paper Code MATH-M2700J)

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January 2017, 2 hours 30 minutes

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*This paper contains **four** questions.*

*Answers to all **FOUR** questions will be used for assessment.*

*On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.*

*Calculators are **not** permitted in this examination.*

*Do not turn over until instructed.*

1. (a) (2+2+3=7 marks) Suppose that  $K \subseteq L$  are fields.
    - (i) Define what it means for the element  $\alpha \in L$  to be *algebraic* over  $K$ , and what it means for the element  $\beta \in L$  to be *transcendental* over  $K$ .
    - (ii) Suppose that  $M$  is a field satisfying  $K \subseteq M \subseteq L$ . Define what it means for  $M : K$  to be a *simple* field extension.
    - (iii) Suppose that  $\gamma \in L$  is transcendental over  $K$ . Define what is meant by the field  $K(\gamma)$ , and give an explicit description of its elements.
  - (b) (7 marks) Suppose that  $\alpha \in L$  is algebraic over  $K$  but does not lie in  $K$ , and  $\beta \in L$  is transcendental over  $K$ . Show that  $K(\alpha, \beta) : K$  is *not* a simple extension.
  - (c) (3+3+5=11 marks)
    - (i) Define what it means for a field extension  $L : K$  to be an *algebraic closure*.
    - (ii) Let  $L : K$  be a field extension with  $K \subseteq L$ , and suppose that  $\alpha \in L$  is algebraic over  $K$ . Define what is meant by the *minimal polynomial* of  $\alpha$  over  $K$ .
    - (iii) Suppose that  $L$  is an algebraically closed field, and that  $M : L$  is an algebraic extension with  $L \subseteq M$ . Show that  $M = L$ .
2. (a) (5 marks) Suppose that  $L : \mathbb{Q}$  is a field extension with  $\mathbb{Q} \subseteq L$ , and that  $\alpha \in L$  is a root of the polynomial  $f(t) = 2t^9 - 25t^3 + 5$ . Prove that there exists no element  $\beta \in \mathbb{Q}(\alpha)$  having minimal polynomial  $m_\beta(\mathbb{Q})$  over  $\mathbb{Q}$  of degree 5.
  - (b) (2+5=7 marks) Suppose that  $E : K$  and  $F : K$  are finite extensions having the property that  $K, E$  and  $F$  are all contained in a field  $L$ .
    - (i) Define what is meant by the *compositum*  $EF$  of the fields  $E$  and  $F$ .
    - (ii) Suppose that  $E : K$  and  $F : K$  are normal. Prove that  $EF : K$  is normal.
  - (c) (2+6=8 marks) Suppose that  $L : M$  is an algebraic field extension with  $M \subseteq L$ , and let  $\overline{M}$  denote an algebraic closure of  $M$ .
    - (i) What does it mean for a polynomial  $f \in M[t]$  to be *separable* over  $M$ ?
    - (ii) Prove that when  $\alpha \in L$  and  $\sigma : M \rightarrow \overline{M}$  is a homomorphism, then  $\sigma(m_\alpha(M))$  is separable over  $\sigma(M)$  if and only if  $m_\alpha(M)$  is separable over  $M$ . [Here we have written  $m_\alpha(M)$  for the minimal polynomial of  $\alpha$  over  $M$ .]
  - (d) (1+4=5 marks) Let  $L : K$  be a field extension with  $K \subseteq L$ , and suppose that  $K$  has characteristic  $p > 0$ .
    - (i) Define the Frobenius monomorphism  $\varphi$  on  $L$ .
    - (ii) Suppose that  $f \in L[t]$  is an inseparable polynomial fixed under the action of the Frobenius map, so that  $\varphi(f) = f$ . Is it possible that  $f \in L[t^p]$ ? Justify your answer.

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3. (a) (10 marks)

Define what it means for a field extension  $L : K$  to be

- (i) normal
- (ii) a splitting field extension
- (iii) an extension by radicals
- (iv) cyclic
- (v) Galois

(b) (15 marks)

Indicate whether each of the following statements is always true or can be false. For those that are false, please provide a short (one or two sentence) justification. Each fully correct answer is worth 1 mark.

(i) When  $L : K$  is an algebraic field extension with  $K \subseteq L$ , and  $L$  is algebraically closed, then the field extension  $L : K$  is normal.

(ii) If  $L : K$  is an algebraic extension of fields with  $K \subseteq L$ , then the algebraic closure  $\overline{L}$  of  $L$  is isomorphic to the algebraic closure  $\overline{K}$  of  $K$ .

(iii) All simple extensions of  $\mathbb{Q}$  are isomorphic.

(iv) There is a homomorphism of finite fields  $\varphi : \mathbb{F}_7 \rightarrow \mathbb{F}_{2017}$ .

(v) Suppose that  $K$  is a field of characteristic  $p$ . If  $L : K$  is a field extension and  $\tau^p \in L$  is transcendental over  $K$ , then  $\tau$  is transcendental over  $K$ .

(vi) An algebraic extension of  $\mathbb{Q}$  has only finitely many subfields.

(vii) Any algebraic extension of  $\mathbb{Q}$  is normal.

(viii) When  $L : K$  is a field extension with  $K \subseteq L$ , and  $\alpha$  and  $\beta$  are distinct elements of  $L$  having the same minimal polynomial over  $K$ , then  $K(\alpha)$  is isomorphic to  $K(\beta)$ .

(ix) The real number  $\sqrt[3]{2 + \sqrt{2}}$  can be constructed by ruler and compass.

(x) Suppose that  $L : K$  is finite and separable. Then  $L : K$  is simple.

(xi) There is no polynomial  $f$  over a finite field of characteristic  $p$  which is irreducible and of degree  $p$ .

(xii) There exist polynomials  $f \in \mathbb{Q}[t]$  of degree 5 solvable by radicals.

(xiii) Suppose that  $M : L$  and  $L : K$  are finite separable extensions. Then  $M : K$  is separable.

(xiv) Suppose that  $M : L$  and  $L : K$  are field extensions with  $M : K$  normal. Then  $M : L$  is a normal field extension.

(xv) Let  $K = \overline{\mathbb{F}_p(t)}$  denote the algebraic closure of  $\mathbb{F}_p(t)$ . Then there exist inseparable polynomials in  $K[X]$ .

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4. (a) (4 marks) State the Fundamental Theorem of Galois Theory.
- (b) (4+4+4 marks) Let  $L : \mathbb{Q}$  be a splitting field extension for  $f(X) = X^4 - 4$ .
- (i) Determine the degree of the extension  $L : \mathbb{Q}$ , justifying your answer.
  - (ii) Describe the Galois group  $\text{Gal}(L : \mathbb{Q})$  (that is, give generators and relations for the Galois group).
  - (iii) Apply the Fundamental Theorem of Galois Theory to find all fields  $M$  for which  $\mathbb{Q} \subsetneq M \subsetneq L$ , explaining carefully how you applied the Fundamental Theorem in this process.
- (c) (3+3+3=9 marks) Let  $L : K$  be a finite Galois extension with Galois group  $G$ , and suppose that  $\alpha \in L$ .
- (i) Let  $f_\alpha(t) = \prod_{\sigma \in G} (t - \sigma(\alpha))$ . Show that  $f_\alpha \in K[t]$ .
  - (ii) Show that the minimal polynomial  $m_\alpha(K)$  of  $\alpha$  over  $K$  divides  $f_\alpha(t)$ .
  - (iii) Suppose that  $\sigma(\alpha) \neq \tau(\alpha)$  whenever  $\sigma, \tau \in G$  satisfy  $\sigma \neq \tau$ . Show that one has  $m_\alpha(K) = f_\alpha$ .

*End of examination.*