UNIVERSITY OF BRISTOL

School of Mathematics

Examination for the Degree of B.Sc. and M.Sci. (Level M)

GALOIS THEORY

MATH M2700

(Paper Code MATH-M2700J)

January 2017, 2 hours 30 minutes

This paper contains **four** *questions.*
Answers to all **FOUR** *questions will be used for assessment.*

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

Calculators are **not** *permitted in this examination.*

Do not turn over until instructed.

1. (a) (2+2+3=7 marks) Suppose that $K \subseteq L$ are fields.

   (i) Define what it means for the element $\alpha \in L$ to be *algebraic* over K, and what it means for the element $\beta \in L$ to be *transcendental* over K.

   (ii) Suppose that M is a field satisfying $K \subseteq M \subseteq L$. Define what it means for $M : K$ to be a *simple* field extension.

   (iii) Suppose that $\gamma \in L$ is transcendental over K. Define what is meant by the field $K(\gamma)$, and give an explicit description of its elements.

   (b) (7 marks) Suppose that $\alpha \in L$ is algebraic over K but does not lie in K, and $\beta \in L$ is transcendental over K. Show that $K(\alpha, \beta) : K$ is *not* a simple extension.

   (c) (3+3+5=11 marks)

   (i) Define what it means for a field extension $L : K$ to be an *algebraic closure*.

   (ii) Let $L : K$ be a field extension with $K \subseteq L$, and suppose that $\alpha \in L$ is algebraic over K. Define what is meant by the *minimal polynomial* of α over K.

   (iii) Suppose that L is an algebraically closed field, and that $M : L$ is an algebraic extension with $L \subseteq M$. Show that $M = L$.

2. (a) (5 marks) Suppose that $L : \mathbb{Q}$ is a field extension with $\mathbb{Q} \subseteq L$, and that $\alpha \in L$ is a root of the polynomial $f(t) = 2t^9 - 25t^3 + 5$. Prove that there exists no element $\beta \in \mathbb{Q}(\alpha)$ having minimal polynomial $m_\beta(\mathbb{Q})$ over $\mathbb{Q}$ of degree 5.

   (b) (2+5=7 marks) Suppose that $E : K$ and $F : K$ are finite extensions having the property that K, E and F are all contained in a field L.

   (i) Define what is meant by the *compositum* EF of the fields E and F.

   (ii) Suppose that $E : K$ and $F : K$ are normal. Prove that $EF : K$ is normal.

   (c) (2+6=8 marks) Suppose that $L : M$ is an algebraic field extension with $M \subseteq L$, and let $\overline{M}$ denote an algebraic closure of M.

   (i) What does it mean for a polynomial $f \in M[t]$ to be *separable* over M?

   (ii) Prove that when $\alpha \in L$ and $\sigma : M \to \overline{M}$ is a homomorphism, then $\sigma(m_\alpha(M))$ is separable over $\sigma(M)$ if and only if $m_\alpha(M)$ is separable over M. [Here we have written $m_\alpha(M)$ for the minimal polynomial of α over M.]

   (d) (1+4=5 marks) Let $L : K$ be a field extension with $K \subseteq L$, and suppose that K has characteristic $p > 0$.

   (i) Define the Frobenius monomorphism φ on L.

   (ii) Suppose that $f \in L[t]$ is an inseparable polynomial fixed under the action of the Frobenius map, so that $\varphi(f) = f$. Is it possible that $f \in L[t^p]$? Justify your answer.

Continued...

3. (a) (10 marks)
Define what it means for a field extension $L : K$ to be

(i) normal
(ii) a splitting field extension
(iii) an extension by radicals
(iv) cyclic
(v) Galois

(b) (15 marks)
Indicate whether each of the following statements is always true or can be false. For those that are false, please provide a short (one or two sentence) justification. Each fully correct answer is worth 1 mark.

(i) When $L : K$ is an algebraic field extension with $K \subseteq L$, and L is algebraically closed, then the field extension $L : K$ is normal.

(ii) If $L : K$ is an algebraic extension of fields with $K \subseteq L$, then the algebraic closure $\overline{L}$ of L is isomorphic to the algebraic closure $\overline{K}$ of K.

(iii) All simple extensions of $\mathbb{Q}$ are isomorphic.

(iv) There is a homomorphism of finite fields $\varphi : \mathbb{F}_7 \to \mathbb{F}_{2017}$.

(v) Suppose that K is a field of characteristic p. If $L : K$ is a field extension and $\tau^p \in L$ is transcendental over K, then τ is transcendental over K.

(vi) An algebraic extension of $\mathbb{Q}$ has only finitely many subfields.

(vii) Any algebraic extension of $\mathbb{Q}$ is normal.

(viii) When $L : K$ is a field extension with $K \subseteq L$, and α and β are distinct elements of L having the same minimal polynomial over K, then $K(\alpha)$ is isomorphic to $K(\beta)$.

(ix) The real number $\sqrt[3]{2 + \sqrt{2}}$ can be constructed by ruler and compass.

(x) Suppose that $L : K$ is finite and separable. Then $L : K$ is simple.

(xi) There is no polynomial f over a finite field of characteristic p which is irreducible and of degree p.

(xii) There exist polynomials $f \in \mathbb{Q}[t]$ of degree 5 solvable by radicals.

(xiii) Suppose that $M : L$ and $L : K$ are finite separable extensions. Then $M : K$ is separable.

(xiv) Suppose that $M : L$ and $L : K$ are field extensions with $M : K$ normal. Then $M : L$ is a normal field extension.

(xv) Let $K = \overline{\mathbb{F}_p(t)}$ denote the algebraic closure of $\mathbb{F}_p(t)$. Then there exist inseparable polynomials in $K[X]$.

Continued...

4. (a) (4 marks) State the Fundamental Theorem of Galois Theory.

   (b) (4+4+4 marks) Let $L : \mathbb{Q}$ be a splitting field extension for $f(X) = X^4 - 4$.

   (i) Determine the degree of the extension $L : \mathbb{Q}$, justifying your answer.

   (ii) Describe the Galois group $\mathrm{Gal}(L : \mathbb{Q})$ (that is, give generators and relations for the Galois group).

   (iii) Apply the Fundamental Theorem of Galois Theory to find all fields M for which $\mathbb{Q} \subsetneq M \subsetneq L$, explaining carefully how you applied the Fundamental Theorem in this process.

   (c) (3+3+3=9 marks) Let $L : K$ be a finite Galois extension with Galois group G, and suppose that $\alpha \in L$.

   (i) Let $f_\alpha(t) = \prod_{\sigma \in G}(t - \sigma(\alpha))$. Show that $f_\alpha \in K[t]$.

   (ii) Show that the minimal polynomial $m_\alpha(K)$ of α over K divides $f_\alpha(t)$.

   (iii) Suppose that $\sigma(\alpha) \neq \tau(\alpha)$ whenever $\sigma, \tau \in G$ satisfy $\sigma \neq \tau$. Show that one has $m_\alpha(K) = f_\alpha$.

End of examination.