ANALYSIS 1A: FIRST ASSESSED HOMEWORK SOLUTIONS

Deadline: To be handed in by 2 pm on Friday October 27th

Hand in: You should hand in your work along with a completed cover sheet in to the marked cabinet on the ground floor of the School of mathematics. Your work should be stapled together with the cover page at the front. Cover sheets are available from the ground floor of the mathematics building, the unit website and blackboard. Solutions will be released online by the 10th of November.

Assessment: This homework will count for 5% of your total mark for Analysis 1A.
Collaboration: The work you hand in should be your own work. You are

welcome to discuss the problems other students but the solutions you hand in should be written solely by you.

Questions

1. **(3+3+3 marks)**
   (a) Find all $x \in \mathbb{R}$ so that $|x-7| \leq 2$.
   (b) Find all $x \in \mathbb{R}$ so that $|x-3| \geq 5$.
   (c) Find all $x \in \mathbb{R}$ with $x \neq -2$ so that $\left|\frac{x-1}{x+2}\right| < 1$.

   Solutions: (a) We have

   $$|x-7| \leq 2 \iff -2 \leq x-7 \leq 2 \iff 5 \leq x \leq 9.$$

   So $|x-7| \leq 2$ exactly when $5 \leq x \leq 9$.

   (b) Case 1. Suppose that $x \geq 3$; so $|x-3| = x-3$. Then $|x-3| \geq 5$ when $x-3 \geq 5$, or equivalently, $x \geq 8$.

   Case 2. Suppose that $x < 3$; so $|x-3| = 3-x$. The $|x-3| \geq 5$ when $3-x \geq 5$, or equivalently, $-2 \geq x$.

   Thus $|x-3| \geq 5$ exactly when either (1) $x \geq 3$ and $x \geq 8$, or (2) $x < 3$ and $x \leq -2$. Hence $|x-3| \geq 5$ exactly when either $x \geq 8$ or $x \leq -2$.

(c) We have

$$\left|\frac{x-1}{x+2}\right| < 1 \iff |x-1| < |x+2|.$$

Case 1. Suppose that $x \geq 1$. Then $|x-1| = x-1$, $|x+2| = x+2$, and $x-1 < x+2 \iff -1 < 2$. Since $-1 < 2$, we have $|x-1| < |x+2|$ whenever $x \geq 1$.

Case 2. Suppose that $-2 < x < 1$. Then $|x-1| = 1-x$, $|x+2| = x+2$, and $1-x < x+2 \iff -1 < 2x$. So when $-2 < x < 1$, we have $|x-1| < |x+2|$ exactly when $-1/x < x$.

Case 3. Suppose that $x < -2$. Then $|x-1| = 1-x$, $|x+2| = -x-2$, and $|x-1| < |x+2| \iff 1 < -2$. As it is not true that $1 < -2$, we never have $|x-1| < |x+2|$ when $x < -2$.

So we have $\left|\frac{x-1}{x+2}\right|$ exactly when either (1) $x \geq 1$, or (2) $-2 < x < 1$ and $-1/2 < x$; more simply, $\left|\frac{x-1}{x+2}\right|$ exactly when $-1/2 < x$.

ALTERNATIVELY: We have

$$\left|\frac{x-1}{x+2}\right| \iff |x-1| < |x+2| \iff (x-1)^2 < (x+2)^2$$

$$\iff 2x-1 < 4x+4 \iff -3 < 6x \iff -1/2 < x.$$

2. **(4+4 marks)**

(a) Let

$$A = \left\{ \frac{n}{n+2} \;:\; n \in \mathbb{N} \right\}.$$

Find $\inf A$, and justify your answer.

(b) Let

$$B = \left\{ \frac{n^2}{n+1} \;:\; n \in \mathbb{Z},\ n \neq -1 \right\}.$$

Show that $\sup B = \infty$.

Solutions: (a) We claim that $\inf A = \frac{1}{3}$. First, we show that $\frac{1}{3}$ is a lower bound for A. To do this, take an arbitrary element of A; so this element is of the form $\frac{n}{n+2}$ where $n \in \mathbb{N}$. So we have $1 \leq n$, hence $2 \leq 2n$, and $n+2 \leq 3n$. Since $3(n+2) > 0$, we get $\frac{1}{3} \leq \frac{n}{n+2}$, showing that $\frac{1}{3}$ is a lower bound for A. With $n = 1 \in \mathbb{N}$, we have $\frac{n}{n+2} = \frac{1}{3}$, so $\frac{1}{3} \in A$. Thus for any $y > \frac{1}{3}$, y cannot be a lower bound for A. Thus $\frac{1}{3}$ is the greatest lower bound for A, meaning that $\inf A = \frac{1}{3}$.

(b) Take $x \in \mathbb{R}$. As $2x \in \mathbb{R}$, by the Archimedean Principle, we can choose $n \in \mathbb{N}$ so that $n \geq 2x$. We know $0 < n+1 < 2n$ so

$$\frac{1}{n+1} > \frac{1}{2n}, \text{ and so } \frac{n^2}{n+1} > \frac{n^2}{2n} = \frac{n}{2} \geq x.$$

As $\frac{n^2}{n+1} \in A$ and $\frac{n^2}{n+1} > x$, x is not an upper bound for B. As this argument holds for all $x \in \mathbb{R}$, B does not have an upper bound in $\mathbb{R}$. This means that $\sup A = \infty$.

3. **(2+2+4 marks)**

(Here you complete part of the proof of Proposition 2.8. You are only to use properties (A1)-(A11), (O1)-(O4), the Completeness Axiom, and the Archimedean Principle. **In particular, given $x \in \mathbb{R}$ with $x > 0$, you are not to assume that $\sqrt{x}$ exists,** as this is what we are trying to prove in Proposition 2.8.)

Suppose that $x, y \in \mathbb{R}$ with $x, y > 0$ and $y^2 > x$.

(a) Suppose that $\varepsilon \in \mathbb{R}$ with $0 < \varepsilon < y$ and $x + 2y\varepsilon < y^2$. Use this inequality to find $\alpha \in \mathbb{R}$ so that $\varepsilon < \alpha$.

(b) With α as in (a), find $\varepsilon \in \mathbb{R}$ with $0 < \varepsilon < y$ and $\varepsilon < \alpha$. Clearly explain your reasoning, using complete sentences.

(c) With ε as in (b), clearly explain why $x + 2y\varepsilon < y^2$, and from this deduce that $x < (y-\varepsilon)^2$.

Solutions: (a) [Full marks for any correct answer, but to help with part (b), α should be independent of ε.] Given the assumptions, we have $0 < 2y\varepsilon < y^2 - x$. So $0 < \varepsilon < \frac{y^2-x}{2y}$. Hence with $\alpha = \frac{y^2-x}{2y}$, we have $\varepsilon < \alpha$.

(b) [Full marks for any correct answer.] Here $\alpha = \frac{y^2-x}{2y}$. Take $\varepsilon = \min(y/2, \alpha/2)$, meaning that ε is the smaller of the values $y/2$ and $\alpha/2$. As $y/2, \alpha/2 > 0$, we have $0 < \varepsilon < y$.

(c) We have $\varepsilon \leq \frac{\alpha}{2} < \alpha = \frac{y^2-x}{2y}$, so $2y\varepsilon < y^2 - x$. Hence

$$x < y^2 - 2y\varepsilon = (y-\varepsilon)^2 - \varepsilon^2 < (y-\varepsilon)^2.$$

[**Note:** These are steps in showing that with x, y positive real numbers, if for every ε with $0 < \varepsilon < y$ we have $(y-\varepsilon)^2 \leq x$, then $y^2 \leq x$. To prove this statement, for the sake of contradiction, we assume that for every ε with $0 < \varepsilon < y$ we have $(y-\varepsilon)^2 \leq x$, but that $y^2 > x$. We aim to find a contradiction by using that $y^2 > x$ to produce some ε with $0 < \varepsilon < y$ but $(y-\varepsilon)^2 > x$. So first we note that

$$(y-\varepsilon)^2 = y^2 - 2y\varepsilon + \varepsilon^2 > y^2 - 2\varepsilon y.$$

So if we can find ε so that $y^2 - 2y\varepsilon > x$, then we get $(y-\varepsilon)^2 > x$. Then we note that since y is positive, we have

$$\begin{aligned} y^2 - 2y\varepsilon > x &\iff x + 2y\varepsilon < y^2 \\ &\iff \varepsilon < \frac{y^2 - x}{2y} \end{aligned}$$

(this is part (a)). With the assumptions that $y^2 > x$ and $y > 0$, we know that $\frac{y^2-x}{2y} > 0$. So if we can find ε so that $0 < \varepsilon < y$ and $\varepsilon < \frac{y^2-x}{2y}$, then we can show that with this choice of ε we have $(y-\varepsilon)^2 > x$, contradicting one of our assumptions. So taking $\alpha = \frac{y^2-x}{2y}$ and $\varepsilon = \min(y/2, \alpha/2)$ we have $0 < \varepsilon < y$, and $\varepsilon < \alpha$ (this is part (b)). Hence we have $\varepsilon < \alpha = (y^2 - x)/(2y)$, so $x < y^2 - 2y\varepsilon < (y-\varepsilon)^2$ (this is part (c)).]