ANALYSIS 1A: SECOND ASSESSED HOMEWORK
Solutions

Deadline: To be handed in by 2 pm on Friday November 24th

Hand in: You should hand in your work along with a completed cover sheet to the marked cabinet on the ground floor of the School of mathematics. Your work should be stapled together with the cover page at the front. Cover sheets are available from the ground floor of the mathematics building, the unit website and Blackboard. Solutions will be released online by the 1st of December.

Assessment: This homework will count for 5% of your total mark for Analysis 1A.

Collaboration: The work you hand in should be your own work. You are welcome to discuss the problems other students but the solutions you hand in should be written solely by you.

Modifications to original Lecture Notes: We proved a strengthened version of Proposition 4.6, as follows.

Proposition 4.6. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers and $\alpha \in \mathbb{R}$. The following statements are equivalent.*

(1) For all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$, we have

$$|a_n - \alpha| \leq \epsilon.$$

(2) Take $K > 0$; for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$, we have

$$|a_n - \alpha| \leq K\epsilon.$$

(3) For all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$, we have

$$|a_n - \alpha| < \epsilon.$$

Following Definition 5.5, we made the following remarks.

Remark. *Suppose that $(n_k)_{k\in\mathbb{N}}$ is a strictly monotone increasing sequence of natural numbers. Then (as one can prove by induction) for every $k \in \mathbb{N}$ we have $n_k \geq k$. Also, if $(a_{n_k})_{k\in\mathbb{N}}$ is a subsequence of a bounded sequence $(a_n)_{n\in\mathbb{N}}$, then $(a_{n_k})_{k\in\mathbb{N}}$ is also a bounded sequence.*

Questions

1. **(2+2+4=8 marks)**

   (a) For all $n \in \mathbb{N}$, let $a_n = \dfrac{1}{7n+9}$. Use Theorem 4.7 (the Sandwich Rule) to evaluate $\lim_{n\to\infty} a_n$. Indicate how you are using the theorem to obtain your result.

   (b) For all $n \in \mathbb{N}$, let $b_n = \dfrac{7n^2 - 5n + 1}{3n^2 - 4}$. Use Theorem 4.9 (rules for limits) and any other relevant results from Section 4 of the Lecture Notes to evaluate $\lim_{n\to\infty} b_n$.

   (c) (Here you give a different proof of Proposition 4.11 (3).) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers so that $\lim_{n\to\infty} a_n = 0$ and for all $n \in \mathbb{N}$, $a_n < 0$. For all $n \in \mathbb{N}$, set $b_n = \dfrac{1}{a_n}$. Working directly from the definition of a divergent sequence (Definition 4.10), show that $\lim_{n\to\infty} b_n = -\infty$. (Suggestion: Begin by showing that for $a, x \in \mathbb{R}$, $\dfrac{1}{a} < x < 0 \iff 0 < -a < -\dfrac{1}{x}$.)

Solutions: (a) For any $n \in \mathbb{N}$, we have $7n + 9 > n > 0$ and so $0 < \dfrac{1}{7n+9} < \dfrac{1}{n}$. Since $\lim_{n\to\infty} 0 = 0$ and $\lim_{n\to\infty} \frac{1}{n} = 0$, by Theorem 4.7 we must have $\lim_{n\to\infty} \frac{1}{7n+9} = 0$.

(b) We have

$$\frac{7n^2 - 5n + 1}{3n^2 - 4} = \frac{7 - 5/n + 1/n^2}{3 - 4/n^2}.$$

By Theorem 4.9 and Corollary 4.8, we have

$$\begin{aligned}
\lim_{n\to\infty}(7 - 5/n + 1/n^2) &= \lim_{n\to\infty} 7 \; - 5 \cdot \lim_{n\to\infty} \frac{1}{n} \; + \; \lim_{n\to\infty} \frac{1}{n^2} \\
&= 7 - 5 \cdot 0 + 0 \\
&= 7
\end{aligned}$$

and

$$\begin{aligned}
\lim_{n\to\infty}(3 - 4/n^2) &= \lim_{n\to\infty} 3 \; - 4 \cdot \lim_{n\to\infty} \frac{1}{n^2} \\
&= 3 - 4 \cdot 0 \\
&= 3.
\end{aligned}$$

Then again using Theorem 4.9, we get

$$\lim_{n\to\infty} \frac{7n^2 - 5n + 1}{3n^2 - 4} = \frac{7}{3}.$$

(c) First note that for $a, x \in \mathbb{R}$, we have

$$\frac{1}{a} < x < 0 \iff \frac{1}{x} < a < 0$$

[since $\frac{a}{x} > 0$ when $\frac{1}{a} < x < 0$], and

$$\frac{1}{x} < a < 0 \iff -\frac{1}{x} > -a > 0.$$

Take $x \in \mathbb{R}$. If $x \geq 0$ then for all $n \in \mathbb{N}$, we have $b_n = \frac{1}{a_n} < x$ [as a_n and hence b_n are negative for all $n \in \mathbb{N}$].

Suppose now that $x < 0$. [Scratch work: We want to find $N \in \mathbb{N}$ so that for all $n \in \mathbb{N}$ with $n \geq N$, we have $b_n = \frac{1}{a_n} < x$. From above, we know that $\frac{1}{a_n} < x < 0 \iff 0 < -a_n < -\frac{1}{x}$. Also, $-a_n = |a_n|$ for all $n \in \mathbb{N}$.] Take $\varepsilon = -\frac{1}{x}$; since $x < 0$, we know $\varepsilon > 0$. Since $\lim_{n\to\infty} a_n = 0$ we know that there is some $N \in \mathbb{N}$ so that for $n \in \mathbb{N}$ with $n \geq N$ we have $|a_n| < \varepsilon$. Thus for $n \geq N$, we have $0 < -a_n < -\frac{1}{x}$, and so $b_n = \frac{1}{a_n} < x < 0$.

[Thus for any $x \in \mathbb{R}$, there is some $N \in \mathbb{N}$ so that for all $n \in \mathbb{N}$ with $n \geq N$ we have $b_n < x$.] Hence by definition, we have $\lim_{n\to\infty} b_n = -\infty$.

2. **(3+3+2=8 marks)** For all $n \in \mathbb{N}$, set $a_n = \dfrac{n}{n+3}$.

   (a) Show that $(a_n)_{n\in\mathbb{N}}$ is a bounded, monotone sequence.

   (b) Show that for $m, n \in \mathbb{N}$, we have

   $$|a_n - a_m| \leq \frac{3}{n} + \frac{3}{m}.$$

   (c) Show that $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Solutions: (a) To show that $(a_n)_{n\in\mathbb{N}}$ is a bounded sequence, we first note that for all $n \in \mathbb{N}$ we have $a_n = \frac{n}{n+3} > 0$. Also, for $n \in \mathbb{N}$, we have $0 < n < n + 3$ and so [remembering that $n + 3 > 0$] $\frac{n}{n+3} < 1$. So for all $n \in \mathbb{N}$, we have $0 < a_n < 1$, showing that $(a_n)_{n\in\mathbb{N}}$ is bounded.

To show that $(a_n)_{n\in\mathbb{N}}$ is monotone, take $n \in \mathbb{N}$. Then $n^2 + 4n \leq n^2 + 4n + 3$, so $n(n + 4) \leq (n + 1)(n + 3)$ and hence $\frac{n}{n+3} \leq \frac{n+1}{(n+1)+3}$. Thus $a_n \leq a_{n+1}$.

(b) Take $m, n \in \mathbb{N}$. Then

$$\begin{aligned}
|a_n - a_m| &= \left| \frac{n}{n+3} - \frac{m}{m+3} \right| \\
&= \left| \frac{n+3-3}{n+3} - \frac{m+3-3}{m+3} \right| \\
&= \left| -\frac{3}{n+3} + \frac{3}{m+3} \right|.
\end{aligned}$$

So first using the Triangle Inequality, we have

$$|a_n - a_m| \leq \left| -\frac{3}{n+3} \right| + \left| \frac{3}{m+3} \right| = \frac{3}{n+3} + \frac{3}{m+3} < \frac{3}{n} + \frac{3}{m}.$$

[Alternatively:

$$\begin{aligned}
|a_n - a_m| &= \left| \frac{n(m+3) - m(n+3)}{(n+3)(m+3)} \right| = \left| \frac{3(n-m)}{(n+3)(m+3)} \right| \\
&< \left| \frac{3(n-m)}{nm} \right| = \left| \frac{3}{m} - \frac{3}{n} \right| \\
&\leq \frac{3}{m} + \frac{3}{n},
\end{aligned}$$

giving us the inequality desired.]

(c) Choose $\varepsilon > 0$. By the Archimedean Principle, there is some $N \in \mathbb{N}$ so that $N \geq \frac{6}{\varepsilon}$. Thus for $m, n \in \mathbb{N}$ with $m, n \geq N$ we have $m \geq \frac{6}{\varepsilon}$ and $n \geq \frac{6}{\varepsilon}$, so [remembering that $\frac{6}{\varepsilon} > 0$] we get $\frac{3}{m} \leq \frac{\varepsilon}{2}$ and $\frac{3}{n} \leq \frac{\varepsilon}{2}$. Hence for all $m, n \in \mathbb{N}$ with $m, n \geq N$, we have

$$|a_n - a_m| \leq \frac{3}{m} + \frac{3}{n} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $(a_n)_{n \in \mathbb{N}}$ is a Cauchy sequence.

3. **(4+5=9 marks)** For all $n \in \mathbb{N}$, set $a_n = \dfrac{7 + (-1)^n 3n}{n}$.

   (a) Using the definition of a divergent sequence, show that $(a_n)_{n \in \mathbb{N}}$ is a divergent sequence.

   (b) Find two convergent subsequences $(a_{n_k})_{k \in \mathbb{N}}$ and $(a_{n_j})_{j \in \mathbb{N}}$ of $(a_n)_{n \in \mathbb{N}}$, and using the results of Section 4 of the Lecture Notes, show that these two sequences are indeed convergent. Then use these subsequences and results from Section 5 of the Lecture Notes to show that $(a_n)_{n \in \mathbb{N}}$ diverges. (When using one of the results from Sections 4 and 5, clearly indicate which result you are using.)

Solutions: (a) Suppose $n \in \mathbb{N}$. When n is even we have

$$|a_n - a_{n+1}| = \left|\frac{7}{n} + 3 - \frac{7}{n+1} + 3\right| = 6 + \frac{7}{n(n+1)} > 6.$$

When n is odd we have

$$|a_n - a_{n+1}| = \left|\frac{7}{n} - 3 - \frac{7}{n+1} - 3\right| = 6 - \frac{7}{n(n+1)} \geq 6 - \frac{7}{2} = \frac{5}{2}.$$

Take $\alpha \in \mathbb{R}$, and take $\varepsilon = 1$. Then for all $N \in \mathbb{N}$, we have

$$\frac{5}{2} \leq |a_N - a_{N+1}| = |a_N - \alpha + \alpha - a_{N+1}| \leq |a_N - \alpha| + |a_{N+1} - \alpha|$$

[where the last inequality comes from the Triangle Inequality]. Thus $|a_N - \alpha| > 1$ or $|a_{N+1} - \alpha| > 1$ (else $\frac{5}{2} \leq 1 + 1$). [So for all $\alpha \in \mathbb{R}$, there is some $\varepsilon > 0$ so that for all $N \in \mathbb{N}$, there is some $n \in \mathbb{N}$ with $n \geq N$ and $|a_n - \alpha| > \varepsilon$.] Therefore $(a_n)_{n\in\mathbb{N}}$ does not converge to any $\alpha \in \mathbb{R}$.

[Alternatively, with $\alpha \in \mathbb{R}$, $\varepsilon = 1$ and any $N \in \mathbb{N}$, we have

$$6 < |a_{2N} - a_{2N+1}| \leq |a_{2N} - \alpha| + |a_{2N+1} - \alpha|,$$

and so we can conclude that $|a_{2N} - \alpha| > \varepsilon$ or $|a_{2N+1} - \alpha| > \varepsilon$. Note that this argument holds when we replace ε by any positive value less than or equal to 3.]

(b) For $j, k \in \mathbb{N}$, set $n_k = 2k$ and set $n_j = 2j + 1$ [or $n_j = 2j - 1$]. Thus $a_{n_k} = \frac{7}{2k} + 3$ and $a_{n_j} = \frac{7}{2j+1} - 3$. Using Theorem 4.9 and then Theorem 4.7, we have

$$\lim_{k\to\infty} \left(\frac{7}{2k} + 3\right) = \frac{7}{2} \cdot \lim_{k\to\infty} \frac{1}{k} + \lim_{k\to\infty} 3 = \frac{7}{2} \cdot 0 + 3 = 3.$$

By Proposition 5.6, $\lim_{j\to\infty} \frac{1}{2j+1} = 0$ [since $\lim_{n\to\infty} \frac{1}{n} = 0$]. Then by Theorems 4.9 and 4.7, we have

$$\lim_{j\to\infty} \left(\frac{7}{2j+1} - 3\right) = 7 \cdot \lim_{j\to\infty} \frac{1}{2j+1} - \lim_{j\to\infty} 3 = 7 \cdot 0 - 3 = -3.$$

By Theorem 5.6, $(a_n)_{n\in\mathbb{N}}$ cannot converge to some $\alpha \in \mathbb{R}$, else all its subsequences would also converge to α [and we have seen one subsequence that converges to 3 and another subsequence that converges to -3].