ANALYSIS 1A: SECOND ASSESSED HOMEWORK

Deadline: To be handed in by 2 pm on Friday November 24th

Hand in: You should hand in your work along with a completed cover sheet to the marked cabinet on the ground floor of the School of mathematics. Your work should be stapled together with the cover page at the front. Cover sheets are available from the ground floor of the mathematics building, the unit website and blackboard. Solutions will be released online by the 1st of December.

Assessment: This homework will count for 5% of your total mark for Analysis 1A.

Collaboration: The work you hand in should be your own work. You are welcome to discuss the problems other students but the solutions you hand in should be written solely by you.

Modifications to original Lecture Notes: We proved a strengthened version of Proposition 4.6, as follows.

Proposition 4.6. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers and $\alpha \in \mathbb{R}$. The following statements are equivalent.*

(1) For all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$, we have

$$|a_n - \alpha| \leq \epsilon.$$

(2) Take $K > 0$; for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$, we have

$$|a_n - \alpha| \leq K\epsilon.$$

(3) For all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$, we have

$$|a_n - \alpha| < \epsilon.$$

Following Definition 5.5, we made the following remarks.

Remark. *Suppose that $(n_k)_{k\in\mathbb{N}}$ is a strictly monotone increasing sequence of natural numbers. Then (as one can prove by induction) for every $k \in \mathbb{N}$ we have $n_k \geq k$. Also, if $(a_{n_k})_{k\in\mathbb{N}}$ is a subsequence of a bounded sequence $(a_n)_{n\in\mathbb{N}}$, then $(a_{n_k})_{k\in\mathbb{N}}$ is also a bounded sequence.*

Questions

1. **(2+2+4=8 marks)**

   (a) For all $n \in \mathbb{N}$, let $a_n = \dfrac{1}{7n+9}$. Use Theorem 4.7 (the Sandwich Rule) to evaluate $\lim\limits_{n\to\infty} a_n$. Indicate how you are using the theorem to obtain your result.

   (b) For all $n \in \mathbb{N}$, let $b_n = \dfrac{7n^2-5n+1}{3n^2-4}$. Use Theorem 4.9 (rules for limits) and any other relevant results from Section 4 of the Lecture Notes to evaluate $\lim\limits_{n\to\infty} b_n$.

   (c) (Here you give a different proof of Proposition 4.11 (3).) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers so that $\lim_{n\to\infty} a_n = 0$ and for all $n \in \mathbb{N}$, $a_n < 0$. For all $n \in \mathbb{N}$, set $b_n = \dfrac{1}{a_n}$. Working directly from the definition of a divergent sequence (Definition 4.10), show that $\lim_{n\to\infty} b_n = -\infty$. (Suggestion: Begin by showing that for $a, x \in \mathbb{R}$, $\dfrac{1}{a} < x < 0 \iff 0 < -a < -\dfrac{1}{x}$.)

2. **(3+3+2=8 marks)** For all $n \in \mathbb{N}$, set $a_n = \dfrac{n}{n+3}$.

   (a) Show that $(a_n)_{n\in\mathbb{N}}$ is a bounded, monotone sequence.

   (b) Show that for $m, n \in \mathbb{N}$, we have
   $$|a_n - a_m| \le \frac{3}{n} + \frac{3}{m}.$$

   (c) Show that $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

3. **(4+5=9 marks)** For all $n \in \mathbb{N}$, set $a_n = \dfrac{7+(-1)^n 3n}{n}$.

   (a) Using the definition of a divergent sequence, show that $(a_n)_{n\in\mathbb{N}}$ is a divergent sequence.

   (b) Find two convergent subsequences $(a_{n_k})_{k\in\mathbb{N}}$ and $(a_{n_j})_{j\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$, and using the results of Section 4 of the Lecture Notes, show that these two sequences are indeed convergent. Then use these subsequences and results from Section 5 of the Lecture Notes to show that $(a_n)_{n\in\mathbb{N}}$ diverges. (When using one of the results from Sections 4 and 5, clearly indicate which result you are using.)