UNIVERSITY OF BRISTOL

Examination for the Degree of B.Sc. and M.Sci. (Level C/4)

FOUNDATIONS AND PROOF

MATH 10004

(Paper Code MATH-10004J)

January 2018 1 hour 30 minutes

This paper contains two sections, Section A and Section B. Please use a separate answer booklet for each section.

*Section A contains **five** short questions. **ALL** answers will be used for assessment. This section is worth 40% of the marks for the paper.*

*Section B contains **two** longer questions. **ALL** answers will be used for assessment. This section is worth 60% of the marks for the paper.*

Calculators are **not** *permitted in this examination.*

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

Do not turn over until instructed.

Section A: Short Questions

A1. (2+2+4 marks)

(i) Negate the following statement:

$$(\exists\, m \in \mathbb{Z} \text{ such that } |a_m| \geq 5) \land (\forall\, n \in \mathbb{N}, n > m \implies |a_n| < 5)$$

(ii) State the contrapositive of the following statement:

$$\forall\, a, b \in \mathbb{Z}, \exists\, m \in \mathbb{Z} \text{ such that } b = am \implies a \mid b$$

(iii) Let P and Q be statements. Using a truth table, show that $P \Leftrightarrow Q$ is equivalent to $(P \implies Q) \land (Q \implies P)$.

A2. (4+2+2 marks)

(i) Find $x \in \mathbb{Z}$ such that $x \equiv 2 \pmod 3$ and $x \equiv 4 \pmod 5$.

(ii) Find $x \in \mathbb{Z}$ (with $0 \leq x \leq 6$) such that $x \equiv 3^{42} \pmod 7$.

(iii) Let $x, y \in \mathbb{Z}$. We say that x and y have the same *parity* if either both x and y are odd or both x and y are even. We define a relation on $\mathbb{Z}$ by $x \sim y$ if x and y have the same parity. For $x \in \mathbb{Z}$, list (without proof) all distinct equivalence classes of x.

A3. (2+2+4 marks)

(i) Define the Cartesian Product of $\mathbb{N}$ and $\mathbb{Z}$.

(ii) (a) Give an example of a partition of $\mathbb{Q}$.

(b) Prove that the example found in (a) is indeed a partition.

A4. (3+3+2 marks)

(i) Define $f = \{(x^3, x) : x \in \mathbb{R}\}$. Is f a function? Justify.

(ii) Define $g = \{(a, \alpha), (b, \delta), (a, \gamma), (c, \delta)\}$ where $a, b, c, \alpha, \delta, \gamma$ are all distinct. Is g a function? Justify.

(iii) Let $X = \{x \in \mathbb{R} : x = 2n + 1, \text{ for some } n \in \mathbb{N}\}$. Define $h : X \to \mathbb{R}$ by $h(x) = 2x$. List the domain, co-domain and range of h.

A5. (2+4+2 marks)

(i) Let A, B be sets. Define what it means for A and B to have the same cardinality.

(ii) Let A and B be sets such that $|A| = |B|$. Prove that if A is countable, then B is countable.

(iii) State the Cantor-Schröder-Bernstein Theorem.

Section B: Longer Questions

B1. (i) (2+8 marks)

(a) Define what it means for p to be prime.

(b) Find all primes p such that $p + 4 = n^2$ for some $n \in \mathbb{N}$.

(ii) (2+8 marks)

(a) Let A and B be sets. What does it mean for A to be a subset of B?

(b) Let A, B, and C be sets. Prove that $A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$.

(iii) (4+4+2 marks)
Define $f : \mathbb{R}^+ \to \mathbb{R}^+$ by $f(x) = \frac{2}{|x|}$.

(a) Prove or disprove that f is injective.

(b) Prove or disprove that f is surjective.

(c) Is f bijective? Briefly justify.

B2. (i) (8+2 marks)

(a) Prove by induction that $2^n > n^2$ for all $n \in \mathbb{N}$ with $n \geq 5$.

(b) Find all $n \in \mathbb{N}$ such that $2^n < n^2$.

(ii) (3+3+3+1 marks)
Let $A, B \in \mathscr{P}(\mathbb{Z})$ (where $\mathscr{P}(\mathbb{Z})$ denotes the Power Set of $\mathbb{Z}$). Define a relation on $\mathscr{P}(\mathbb{Z})$ by $A \sim B$ if $A \cap B = \emptyset$.

(a) Prove or disprove that $\sim$ is reflexive.

(b) Prove or disprove that $\sim$ is symmetric.

(c) Prove or disprove that $\sim$ is transitive.

(d) Is $\sim$ an equivalence relation? Briefly justify.

(iii) (2+8 marks)

(a) Let $a, b \in \mathbb{Z}$. What is the relationship between $\gcd(a, b)$ and $\operatorname{lcm}(a, b)$?

(b) Let $a, b, m, n \in \mathbb{N}$ with $a, b \geq 2$. Show that if $\gcd(a, b) = 1$, then $\operatorname{lcm}(a^n, b^m) = a^n b^m$.

End of examination.