UNIVERSITY OF BRISTOL

Examination for the Degree of B.Sc. and M.Sci. (Level M)

GALOIS THEORY

MATH M2700

(Paper Code MATH-M2700J)

January 2018, 2 hours 30 minutes

This paper contains **four** *questions*
All **FOUR** *questions should be attempted.*

Calculators are **not** *permitted in this examination.*

Do not turn over until instructed.

1. Suppose that $L : K$ is a field extension with $K \subseteq L$.

   (a) **(2+1+2+3+3+2+4=17 marks)**

   (i) State what it means for $\alpha \in L$ to be algebraic over K.

   (ii) State what it means for L to be algebraic over K.

   (iii) For $\alpha \in L$, give the definitions of $K[\alpha]$ and $K(\alpha)$.

   (iv) For $\alpha \in L$, state when $K[\alpha] = K(\alpha)$.

   (v) Suppose that $\alpha \in L$ is algebraic over K. Define $m_\alpha(K)$, the minimal polynomial of α over K.

   (vi) Suppose that $0 \neq \alpha \in L$ is algebraic over K, and let a_0 denote the constant term of $m_\alpha(K)$. Briefly explain why $a_0 \neq 0$.

   (vii) Suppose that $0 \neq \alpha \in L$ is algebraic over K. Show that $\alpha^{-1} \in K[\alpha]$.

   (b) **(3+5=8 marks)**

   (i) State the Tower Law.

   (ii) Suppose that $\alpha, \beta \in L$ are algebraic over K. Show that $\alpha + \beta$ is algebraic over K.

2. Suppose that K is a field.

   (a) **(2+5=7 marks)**

   (i) Define what it means for a field to be an algebraic closure of K.

   (ii) Suppose that L is a field so that $K \subseteq L \subseteq \overline{K}$ where $\overline{K}$ denotes an algebraic closure of K. Further, suppose that $\alpha, \beta \in L$ so that there is a K-homomorphism $\tau : K(\alpha) \to K(\beta)$ with $\tau(\alpha) = \beta$. Show that $m_\alpha(K) = m_\beta(K)$.

   (b) **(2+2+4+5=13 marks)**
   Suppose that $K \subseteq \overline{K}$ and $\alpha \in \overline{K}$ where $\overline{K}$ is an algebraic closure of K.

   (i) Define what it means for α to be separable over K.

   (ii) Present a basis for $K(\alpha)$ as a vector space over K.

   (iii) Let $\sigma : K \to \overline{K}$ denote the inclusion map (so for every $\gamma \in K$, we have $\sigma(\gamma) = \gamma$). Precisely describe how to extend σ to a homomorphism $\tau : K(\alpha) \to \overline{K}$ and briefly explain why τ is well-defined. (You do not need to verify that τ is indeed a homomorphism.)

   (iv) Show that the number of ways one can construct such τ (as in (iii)) is $[K(\alpha) : K]$ if α is separable over K, and less than $[K(\alpha) : K]$ if α is not separable over K.

   (c) **(5 marks)** Suppose that p is prime, and $g \in \mathbb{F}_p[t]$ is irreducible of degree $d > 1$. Suppose that $\mathbb{F}_p \subseteq \overline{\mathbb{F}}_p$ and $\alpha \in \overline{\mathbb{F}}_p$ so that α is a root of g. Show that α^p is a root of g and that $\alpha^p \neq \alpha$.

3. (a) **(2+1+2=5 marks)** Suppose that K, L are fields with $K \subseteq L$.

   (i) Define what it means for $f \in K[t] \setminus K$ to split over L.

   (ii) Define what it means for $L : K$ to be a splitting field extension for $f \in K[t] \setminus K$.

   (iii) Define what it means for $L : K$ to be a simple extension.

   (b) **(15+5=20 marks)**

   Indicate whether each of the following statements is always true or can be false (1 mark for each correct answer). For **5** of the statements you identify as false, provide a short justification (1 mark for each correct justification).

   (i) Suppose that $\gamma \in \mathbb{C}$ is transcendental over $\mathbb{Q}$. Then there exist infinitely many fields M with $\mathbb{Q} \subseteq M \subseteq \mathbb{Q}(\gamma)$.

   (ii) There is a normal extension $L : K$ with some $\alpha \in L$ so that $m_\alpha(K)$ does not split over L.

   (iii) Suppose that $L : K$ is a field extension with $K \subseteq L$, $\alpha \in L$, and $E_\alpha : K[t] \to L$ the evaluation map. Then α is transcendental over K if and only if E_α is injective.

   (iv) Suppose that K, M, L are fields with $K \subseteq M \subseteq L$ and $L : K$ a Galois extension. Then $M : K$ is a Galois extension.

   (v) Every finite, separable field extension is simple.

   (vi) With t an indeterminate, $\mathbb{F}_3[t]/(t^2 + t + 1)$ is a field with 9 elements.

   (vii) The polynomial $t^5 + 3$ is separable over $\mathbb{F}_{25}$.

   (viii) $\mathbb{C}$ is an algebraic closure of $\mathbb{Q}$.

   (ix) Suppose that $L : K$ is a field extension with $\gamma \in L$ so that γ is transcendental over K. Then $\gamma^3 + \gamma + 1$ is transcendental over $K(\gamma)$.

   (x) Suppose that $t^n - \alpha \in \mathbb{Q}[t]$ and that $L : \mathbb{Q}$ is a splitting field extension for $t^n - \alpha$. Then L contains a primitive nth root of unity.

   (xi) There is a finite field L with subfields K_1, K_2 so that $[L : K_1] = [L : K_2]$ but K_1 is not isomorphic to K_2.

   (xii) Suppose that $L : K$ is a splitting field extension for $f \in K[t] \setminus K$. If $\alpha \in L$ then $m_\alpha(K)$ splits in $L[t]$.

   (xiii) With K a field, there are normal field extensions $L : K$ and $M : K$ with $K \subseteq L \subseteq \overline{K}$, $K \subseteq M \subseteq \overline{K}$, and $(L \cap M) : K$ not a normal extension.

   (xiv) With K a field and $f \in K[t] \setminus K$, if f has one multiple root in $\overline{K}$ then all the roots of f are multiple roots.

   (xv) Suppose that $L : M : K$ is a tower of field extensions so that $L : K$ is a normal extension. Then $L : M$ is a normal extension.

4. (a) **(5+3+4=12 marks)** Set $f = (t^2 - 5)(t^2 + 3)$, and let $L : \mathbb{Q}$ be a splitting field extension for f with $\mathbb{Q} \subseteq L \subseteq \mathbb{C}$.

   (i) Determine the degree of $L : \mathbb{Q}$, carefully justifying your answer.

   (ii) Describe the Galois group $Gal(L : \mathbb{Q})$ (that is, give generators and relations for the Galois group).

   (iii) Apply the Fundamental Theorem of Galois Theory to find all fields M for which $\mathbb{Q} \subsetneq M \subsetneq L$, explaining carefully how you applied the Fundamental Theorem in this process.

   (b) **(7 marks)** Let $g = (t^2 - 2)(t^3 - 7)$, and let $L : \mathbb{Q}$ be a splitting field extension for g with $\mathbb{Q} \subseteq L \subseteq \mathbb{C}$. Determine the value of $[L : \mathbb{Q}]$, carefully explaining your reasoning.

   (c) **(6 marks)**

   Suppose that $E : \mathbb{Q}$ and $F : \mathbb{Q}$ are finite, normal extensions, with $\mathbb{Q} \subseteq E \subseteq \mathbb{C}$ and $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$. Also suppose that $\varphi : E \to F$ is an isomorphism. Show that $E = F$.