UNIVERSITY OF BRISTOL

Examination for the Degree of B.Sc. and M.Sci. (Level C/4)

INTRODUCTION TO PROOFS

MATH 10010

(Paper Code MATH-10010J)

January 2019 1 hour 30 minutes

This paper contains two sections, Section A and Section B. Please use a separate answer booklet for each section.

Section A contains ***five*** *short questions.* ***ALL*** *answers will be used for assessment. This section is worth 40% of the marks for the paper.*

Section B contains ***two*** *longer questions.* ***ALL*** *answers will be used for assessment. This section is worth 60% of the marks for the paper.*

Calculators are **not** *permitted in this examination.*

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

Do not turn over until instructed.

Section A: Short Questions

A1. (2+2+4 marks)

(i) Negate the following statement:

$$p \in \mathbb{P} \Longrightarrow (d \in \mathbb{N} \text{ such that } d \mid p \Rightarrow d = 1 \vee d = p)$$

(ii) State the contrapositive of the following statement:

$$\forall\, a, b \in \mathbb{Z}, \gcd(a, b) = c \Longrightarrow \exists\, x, y \in \mathbb{Z} \text{ such that } a = cx \wedge b = cy$$

(iii) Let P and Q be statements. Using a truth table, show that $\neg(P \wedge Q)$ is equivalent to $\neg P \vee \neg Q$.

A2. (2+2+2+2 marks)

Define $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ by $f((m, n)) = (m + n, n)$, and define $g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ by $g((m, n)) = (m - n, n)$.

(i) Let $(m, n) \in \mathbb{Z} \times \mathbb{Z}$. Determine $(g \circ f)((m, n))$.

(ii) Let $(m, n) \in \mathbb{Z} \times \mathbb{Z}$. Determine $(f \circ g)((m, n))$.

(iii) What can we conclude from (i) and (ii)? Justify your answer.

(iv) Let $V = \{(2n, 0) : n \in \mathbb{N}\}$. Find $g^{-1}[V]$.

A3. (4+2+2 marks)

(i) Find $x \in \mathbb{Z}$ such that $x \equiv 2 \pmod{10}$ and $x \equiv 3 \pmod{7}$.

(ii) Does 0 divide 0? Justify your answer.

(iii) Let p_1 and p_2 be distinct primes. Determine $\operatorname{lcm}(p_1, p_2)$. Justify your answer.

A4. (2+2+2+2 marks)

(i) Let A and B be sets. What is the Cartesian Product of A and B?

(ii) Let A be a set. What is the power set $\mathscr{P}(A)$ of A?

(iii) Let $A = \{\emptyset\}$ and $B = \{\pi\}$.

(a) Find $\mathscr{P}(A) \times \mathscr{P}(B)$.

(b) Define an equivalence relation $\sim$ on $\mathscr{P}(A)$ by $M \sim N$ if and only if $(M, N) \in \mathscr{P}(A) \times \mathscr{P}(A)$. Find $[\emptyset]_\sim$.

A5. (2+3+3 marks)

(i) State (without proof) which of the following sets have the same cardinality:

$$\mathbb{R}_+, \quad \mathbb{Q}, \quad \{0, 1\}, \quad \mathbb{N}, \quad (2, 3)$$

(ii) Let A and B be sets. Prove that if A and B are countable, then $|A| = |B|$.

(iii) Find a partition P of $\mathbb{N}$ such that $P = \{A, B, C\}$, where $|A| = |B| = |C|$. Justify your answer.

Section B: Longer Questions

Please start a new booklet for Section B.

B1. (i) (4+4+2 marks)

We define the *floor function* $f : \mathbb{R} \to \mathbb{Z}$ by $f(x) = \lfloor x \rfloor = \max\{m \in \mathbb{Z} : m \leq x\}$.

(a) Prove or disprove that f is injective.

(b) Prove or disprove that f is surjective.

(c) Is f bijective? Justify your answer.

(ii) (2+8 marks)

(a) Let $f : X \to Y$ and let $V \subseteq Y$. What is the definition of the inverse image of V under f?

(b) Let $f : X \to Y$ and let $U, V \subseteq Y$. Prove that $f^{-1}[U] \cap f^{-1}[V] = f^{-1}[U \cap V]$.

(iii) (10 marks)

Let R be an equivalence relation on a non-empty set X. Then the *inverse relation* of R is given by $R^{-1} = \{(a, b) : (b, a) \in R\}$. Prove that R^{-1} is an equivalence relation on X.

B2. (i) (2+8 marks)

(a) Let A and B be sets. What does it mean for A and B to have the same cardinality?

(b) Let $f : X \to Y$ be injective, and let $A \subseteq X$ be a non-empty set. Show that $|A| = |f[A]|$.

(ii) (6+4 marks)

(a) Define $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by $f((m, n)) = 5^m 11^n$. Prove that f is injective.

(b) Let $\mathbb{P}$ be the set of all primes. Explain why $\bigcup_{p \in \mathbb{P}} p\mathbb{N} = \mathbb{N} \backslash \{1\}$.

(iii) (2+8 marks)

(a) Let P and Q be statements. State the truth table for $P \iff Q$.

(b) Let $a_1 = a_2 = 1$ and $a_n = a_{n-1} + a_{n-2}$ for all $n \in \mathbb{N}$ with $n \geq 3$. Let $P(n)$ be the following statement:

$$3 \mid n \text{ if and only if } a_n \in 2\mathbb{N}.$$

Prove that $P(n)$ holds for all $n \in \mathbb{N}$.