GALOIS THEORY 2019: HW 2
For assessment: Problems 1, 2, 3
Due by noon Tuesday, week 5 of the term

1. Suppose that $L : K$ is a field extension with $K \subseteq L$, and that $\tau : L \to L$ is a K-homomorphism. Also suppose that $f \in K[t]$ with $\deg f \geq 1$, and that $\alpha \in L$. Show that $f(\alpha) = 0$ if and only if $f(\tau(\alpha)) = 0$.

2. Let M be a field. Show that the following are equivalent:
   (i) M is algebraically closed.
   (ii) Every nonconstant polynomial $f \in M[t]$ factors in $M[t]$ as a product of linear factors.
   (iii) Every irreducible polynomial in $M[t]$ has degree 1.
   (iv) The only algebraic extension of M is M itself.

3. Suppose that L and M are fields with an associated homomorphism $\psi : L \to M$. Show that whenever L is algebraically closed, then $\psi(L)$ is also algebraically closed.

4. [This is a HW problem from years ago; it demonstrates a type of result one can prove with ruler and compass constructions.]
   Set $f(t) = t^7 - 7t^5 + 14t^3 - 7t - 2 \in \mathbb{Q}[t]$. With $g_1 = t - 2$ and $g_3 = t^3 + t^2 - 2t - 1$, one can check that $f = g_1 g_3^2$.
   (a) Show that g_3 is irreducible in $\mathbb{Q}[t]$.
   (b) Using the identity
   $$\cos 7\theta = 64 \cos^7 \theta - 112 \cos^5 \theta + 56 \cos^3 \theta - 7 \cos \theta,$$
   together with the conclusion of part (a), show that the angle $2\pi/7$ is not constructible by ruler and compass. Hence deduce that the regular heptagon is not constructible by ruler and compass.

5. Let $L : K$ be a field extension. Show that $Gal(L : K)$ is a subgroup of $Aut(L)$.

6. Suppose K_1, K_2 are fields and $\sigma : K_1 \to K_2$ is an isomorphism. We extend σ to the isomorphism $\sigma : K_1[t] \to K_2[t]$ by setting $\sigma(t) = t$. Suppose that f is an irreducible element of $K_1[t]$.
   (a) Show that $\sigma(f)$ is an irreducible element of $K_2[t]$.
   (b) Define $\varphi : K_1[t] \to K_2[t]/(\sigma(f))$ by $\varphi(g) = \sigma(g) + (\sigma(f))$. Show that $\ker \varphi = (f)$.