GALOIS THEORY 2019: HW 3 Feedback

For assessment: Problems 1, 2, 3

Due by noon Tuesday, week 7 of the term

Overall, this was mostly well done, although quite a few struggled with #3(a) (a former exam problem).

1. (b) Say $|Gal(L:\mathbb{Q})| = 8$, $\sigma, \tau \in Gal(L:\mathbb{Q})$ with $\mathrm{ord}(\sigma) = 4$, $\mathrm{ord}(\tau) = 2$. To conclude that σ, τ generate $Gal(L:\mathbb{Q})$, you need to show that $\tau \notin \langle\sigma\rangle$ (meaning $\tau \neq \sigma^2$). If one shows that $\tau\sigma = \sigma^2\tau$, then one knows $\tau \notin \langle\sigma\rangle$ as $\langle\sigma\rangle$ is abelian.

   In Galois Theory, writing $\alpha = \sqrt[4]{5}$ really just means that $\alpha^4 = 5$. So it is helpful to specify whether you mean $\alpha \in \mathbb{R}$.

2. Once you have concluded $\tau(L) \subseteq L$, you can use Theorem 3.4 to conclude $\tau(L) = L$ (since τ is a K-homomorphism and $L : K$ is algebraic).

3. (a) To show $\deg f_1 = \deg f_2 = \cdots = \deg f_d$, **first** choose k with $1 < k \leq d$, **then** choose $\beta \in L$ as a root of f_k (recall that $f = f_1 \cdots f_d$ splits over L). With α a root of f_1, Proposition 6.6 says there is some $\tau \in Gal(L:K)$ so that $\tau(\alpha) = \beta$. (Theorem 3.2 tells us there is some K-isomorphism $\sigma : K(\alpha) \to K(\beta)$ so that $\sigma(\alpha) = \beta$, but it takes some work to show that σ can be extended to an element of $Gal(L:K)$.) So $0 = \tau(f_1(\alpha)) = \tau(f_1)(\beta)$ and thus β is a root of $\tau(f_1)$. As f_1 is irreducible over M, from Algebra 2 we know that $\tau(f_1)$ is irreducible over $\tau(M)$. By #2, you know $\tau(M) = M$. Hence, as f_k is monic and irreducible over M with β as a root, $\tau(f_1) = m_\beta(M) = f_k$. Thus $\deg f_1 = \deg \tau(f_1) = \deg f_k$.