GALOIS THEORY 2019: HW 3
For assessment: Problems 1, 2, 3
Due by noon Tuesday, week 7 of the term

1. (a) Let $L:\mathbb{Q}$ be a splitting field extension for $f(X)=(X^2-2)(X^2+7)$.
   (i) Determine the degree of the extension $L:\mathbb{Q}$, justifying your answer.
   (ii) Describe the Galois group $\mathrm{Gal}(L:\mathbb{Q})$ (that is, give generators and relations for the Galois group).

   (b) Let $K:\mathbb{Q}$ be a splitting field extension for $g(X)=X^4-5$.
   (i) Show that $[K:\mathbb{Q}]=8$.
   (ii)) Describe the Galois group $\mathrm{Gal}(K:\mathbb{Q})$.

2. Suppose that $L:K$ is a normal extension with $K\subseteq L\subseteq \overline{L}$ where $\overline{L}$ is an algebraic closure of L.
   (a) Suppose $\tau : L\to \overline{L}$ is a K-homomorphism. Show that $\tau(L)=L$.
   (b) Suppose $M:K$ is a normal extension so that $K\subseteq M\subseteq L$ and $\tau\in Gal(L:K)$. Show that $\tau(M)=M$. (Suggestion: use (a).)

3. Suppose that $L:K$ is a splitting field extension for f where f is a monic, separable, irreducible element of $K[t]$ with $\deg f$ prime. Suppose that M is a field so that $K\subsetneq M\subsetneq L$ and $M:K$ is a normal extension. The goal is to show that f is irreducible over M.
   (a) For the sake of contradiction, suppose that $f=f_1\cdots f_d$ where $d>1$ and $f_1,\dots,f_d$ are monic, irreducible elements of $M[t]$. Show that for each integer k with $1<k\le d$, we have $\deg f_k=\deg f_1$. (Suggestion: first use $Gal(L:K)$ to show that for $1<k\le d$, $\deg f_1=\deg f_k$; in doing this, you may want to use Problem 1.)
   (b) Show that the hypothesis of (a) leads to a contradiction (and hence f is irreducible over M). (Suggestion: first explain why M contains no root of f.)

4. Suppose K is a field, $S\subseteq K[t]$. Suppose that $L:K$ is a splitting field extension for S with $K\subseteq L$, and that $M:K$ is a splitting field extension for S relative to the embedding $\varphi:K\to M$. Assume $L\subseteq \overline{L}$, $M\subseteq\overline{M}$. Set $A=\{\alpha\in\overline{L}:\ f(\alpha)=0 \text{ for some nonconstant } f\in S\ \}$, and $B=\{\beta\in\overline{M}:\ \varphi(f)(\beta)=0 \text{ for some nonconstant } f\in S\ \}$. (So $L=K(A)$ and $M=F(B)$ where $F=\varphi(K)$.)
   (a) Explain why there is an isomorphism $\psi:\overline{L}\to\overline{M}$ that extends φ.
   (b) Show that $\psi(A)=B$.
   (c) Conclude that $\psi(K(A))\simeq F(B)$ (and hence $L\simeq M$ since $K(A)=L$ and $F(B)=M$). [Note that the argument used in the proof of Theorem 5.4 shows that $[L:K]=[M:K]$.]