GALOIS THEORY 2019: HW 4
For assessment: Problems 1, 2, 3
Due by noon Tuesday, week 9 of the term

1. (Here you are to prove Proposition 7.2.) Suppose that $L : M$ is an algebraic field extension, and let $\overline{M}$ be an algebraic closure of M; suppose that $M \subseteq L \subseteq \overline{M}$. Also suppose $\alpha \in L$ (so $m_\alpha(M)$ exists), and suppose $\sigma : M \to \overline{M}$ is a homomorphism. Show that if $m_\alpha(M)$ is separable over M then $\sigma(m_\alpha(M))$ is separable over $\sigma(M)$.
2. (Here you are to prove Corollary 8.6.) Suppose char$K = p > 0$ and K is algebraic over its prime subfield. Then all polynomials in $K[t] \setminus K$ are separable over K.
3. (Here you are to prove Proposition 10.1.) Let K, M, L be fields so that $K \subseteq L$ and $M \subseteq L$. Suppose G and H are subgroups of $Aut(L)$. Prove the following.
   (a) If $K \subseteq M$ then $Gal(L : K) \supseteq Gal(L : M)$.
   (b) If G is a subgroup of H, then $Fix_L(G) \supseteq Fix_L(H)$.
   (c) $K \subseteq Fix_L(Gal(L : K))$.
   (d) $G \subseteq Gal(L : Fix_L(G))$.
   (e) $Gal(L : K) = Gal(L : Fix_L(Gal(L : K)))$.
   (f) $Fix_L(G) = Fix_L(Gal(L : Fix_L(G)))$.
4. (This is Corollary 7.6 (b).) Suppose that $L : K$ is a splitting field extension for $S \subseteq K[t] \setminus K$.
   (a) Show that if $L : K$ is a separable extension then each $f \in S$ is separable over K.
   (b) Show that if each $f \in S$ is separable over K then $L : K$ is a separable extension.
5. (This is Theorem 8.1.) Let $f \in K[t]$, $f \neq 0$, and let $L : K$ be a splitting field extension for f. Show that the following are equivalent:
   (i) f has a multiple root in L.
   (ii) There is some $\alpha \in L$ so that $f(\alpha) = 0 = (Df)(\alpha)$.
   (iii) There is some $g \in K[t]$ so that $\deg g \geq 1$ and g divides both f and Df.
6. (This is part of Theorem 9.1.) Suppose that K is an infinite field, α, β are algebraic over K, and $L : K$ is a splitting field extension for
$$m_\alpha(K) \cdot m_\beta(K).$$
   Suppose that $\varphi_1, \ldots, \varphi_r$ are **distinct** monomorphisms from $K(\alpha, \beta)$ into L that fix K pointwise.
   (a) Show that $f \neq 0$, where
$$f = \prod_{i \neq j} \big((\varphi_i(\alpha) - \varphi_j(\alpha)) \,+\, (\varphi_i(\beta) - \varphi_j(\beta))t\big).$$
   (b) Show that there is some $\delta \in K$ so that $f(\delta) \neq 0$.
   (c) With δ as above, set $\gamma = \alpha + \beta\delta$. Show that for $i \neq j$ $(1 \leq i, j \leq r)$, we have $\varphi_j(\gamma) \neq \varphi_j(\gamma)$.