GALOIS THEORY 2020: HW 1

For assessment: Problems 1, 2, 3, 4

Due by 16:00 Tuesday, 11 February

Please present your solutions in legible, complete sentences.

Here your are to remember that with $K \subseteq L$ fields and $\alpha \in L$, α is algebraic over K if and only if $[K(\alpha) : K] < \infty$.

1. (This is from the 2018 exam.) Suppose that $L : K$ is a field extension with $K \subseteq L$. Suppose that $\alpha, \beta \in L$ are algebraic over K. Show that $\alpha + \beta$ is algebraic over K.

2. Suppose $L : K$ is a field extension with $K \subseteq L$ and $\tau : L \to L$ is a K-homomorphism. Suppose $f \in K[t] \setminus K$ and $\alpha \in L$.
   (a) Clearly explain why $\sigma(f(\alpha)) = 0$ if and only if $f(\alpha) = 0$.
   (b) Clearly explain why $\sigma(f(\alpha)) = f(\sigma(\alpha))$.
   [Note that this shows $f(\alpha) = 0 \iff \sigma(f(\alpha)) = 0 \iff f(\sigma(\alpha)) = 0$.]

3. Suppose $L : K$ is an algebraic (but not necessarily finite) field extension, and take $\alpha \in L$. Let $G = Gal(L : K)$.
   (a) Explain why $m_\alpha(K)$ has finitely many roots in L (recall that $m_\alpha(K)$ exists since $L : K$ is algebraic).
   (b) Show that the G-orbit of α is a finite set. (Recall that the G-orbit of α is $\{\sigma(\alpha) : \sigma \in G\}$.)

4. Suppose $L : K$ is an algebraic (but not necessarily finite) field extension, and with $G = Gal(L : K)$, suppose $|G| < \infty$. Take $\alpha \in L$. Let $\alpha_1, \alpha_2, \dots, \alpha_d$ be the *distinct* elements in the G-orbit of α, and set $R = \{\alpha_1, \alpha_2, \dots, \alpha_d\}$ (note that since $|G|$ is finite, so is R, and thus $d \in \mathbb{Z}_+$). Take $\tau \in G$. Show that $\tau(R) = R$.

5. Let $L : K$ be a field extension with $K \subseteq L$. Let $A \subseteq L$, and let
$$\mathcal{C} = \{C \subseteq A : C \text{ is a finite set}\}.$$
   Show that $K(A) = \cup_{C \in \mathcal{C}} K(C)$, and further that when $[K(C) : K] < \infty$ for all $C \in \mathcal{C}$, then $K(A) : K$ is an algebraic extension.

6. (Part (b)(ii) is part of a problem on the 2018 exam.) Let $F = \mathbb{Z}/p\mathbb{Z}$ where p is prime. Recall that for $a \in F$, $a^p = a$.
   (a) For $n \in \mathbb{Z}_+$, use induction on n to show that for $f = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n \in F[t]$, we have
$$f^p = a_0 + a_1 t^p + a_2 t^{2p} + \cdots + a_n t^{np}.$$
   (b) Suppose that $E : F$ is a field extension with $F \subseteq E$, $[E : F] < \infty$, and $\alpha \in E \setminus F$.
      (i) Briefly explain why α is algebraic over F.
      (ii) Let $f = \mathrm{m}_\alpha(F)$, the minimal polynomial of α over F. Show that α^p is a root of f.

(iii) Suppose that $|E| = p^m$. Show that every element of E is a root of $t^{p^m} - t$. (Suggestion: use that $E^\times$ is a field under multiplication. Also recall that $E^\times = E \setminus \{0\}$.)

7. (This is a problem on the 2015 exam.) Let $L : K$ be a field extension, and suppose that $\gamma \in L$ satisfies the property that $\deg m_\gamma(K) = 7$. Suppose that $h \in K[t]$ is a non-zero cubic polynomial. By noting that γ is a root of the cubic polynomial $g(t) = h(t) - h(\gamma) \in K(h(\gamma))[t]$, show that $[K(h(\gamma)) : K] = 7$.