**GALOIS THEORY 2020: HW 2**

**For assessment: Problems 1, 2, 3**

**Due by 16:00 Tuesday, 25 February**

**Please present your solutions in legible, complete sentences.**

1. (Here you prove Proposition 4.7.) Suppose that $L$ and $M$ are fields and $\psi : L \to M$ is a homomorphism. Show that whenever $L$ is algebraically closed, then $\psi(L)$ is also algebraically closed.

2. (a) (Here you prove Proposition 4.9.) Suppose $L : K$ is an algebraic field extension with $K \subseteq L$. Also suppose $\overline{L}$ is an algebraic closure of $L$. Show that $\overline{L}$ is an algebraic closure of $K$.
   (b) Suppose $K, L$ are fields with $K \subseteq L$. Also suppose $M$ is a field so that every irreducible polynomial in $M[t]$ has degree 1, and $L : M$ is a field extension with $[L : M] > 1$. Show that for $\alpha \in L$ with $\alpha \notin M$, we have $[M(\alpha) : M] = \infty$.

3. (a) Find $L$ so that $L : \mathbb{Q}$ is a splitting field extension for
$$f(t) = (t^2 - 2)(t^2 + 7);$$
   determine $[L : \mathbb{Q}]$, clearly explaining your reasoning.
   (b) With $L$ as in (a), describe $Gal(L : \mathbb{Q})$.

4. (Here you prove Theorem 5.5.) Suppose $K$ is a field, $S \subseteq K[t]$. Suppose that $L : K$ is a splitting field extension for $S$ with $K \subseteq L$, and that $M : K$ is a splitting field extension for $S$ relative to the embedding $\varphi : K \to M$. Assume $L \subseteq \overline{L}$, $M \subseteq \overline{M}$. Set
$$A = \{\alpha \in \overline{L} : \ f(\alpha) = 0 \text{ for some nonconstant } f \in S \},$$
   and
$$B = \{\beta \in \overline{M} : \ \varphi(f)(\beta) = 0 \text{ for some nonconstant } f \in S \}.$$
   (So $L = K(A)$ and $M = F(B)$ where $F = \varphi(K)$.)
   (a) Explain why there is an isomorphism $\psi : \overline{L} \to \overline{M}$ that extends $\varphi$.
   (b) Show that $\psi(A) = B$.
   (c) Conclude that $\psi(K(A)) \simeq F(B)$ (and hence $L \simeq M$ since $K(A) = L$ and $F(B) = M$). [Note that the argument used in the proof of Theorem 5.4 shows that $[L : K] = [M : K]$.]

5. (Here you prove Proposition 5.2) Suppose $L : K$ is a splitting field extension for $f \in K[t] \setminus K$. Assume $K \subseteq L$. Set $K_0 = K$, and for $1 \leq i \leq n$, set $K_i = K(\alpha_1, \ldots, \alpha_i)$. With $d = \deg f$, show that for $1 \leq i \leq n$ we have $[K_i : K_{i-1}] \leq d - i + 1$. Hence conclude that $[L : K] \leq d!$.