ANALYSIS 1A EXERCISES: PART 1 of 2

Analysis 1A exercise sheet 1: Rationals and irrationals

1. Show that if $x \in \mathbb{Q}$, $x \neq 0$ and y is an irrational number then $xy \notin \mathbb{Q}$.
2. Show that it is possible to find irrational numbers x, y where $x+y \in \mathbb{Q}$.
3. Which of the following subsets of $\mathbb{Q}$ are bounded by rationals (justify your answer).
   (a) $A_1 = \mathbb{Z}$, the integers.
   (b) $A_2 = \left\{\frac{n-1}{n} : n \in \mathbb{N}\right\}$.
4. (a) Show that the set $A_1 = \left\{\frac{n+5}{6} : n \in \mathbb{N}\right\}$ is unbounded above in the rationals. (To do this you need to show that for all $x = \frac{p}{q} \in \mathbb{Q}$ there exists $n \in \mathbb{N}$ where $\frac{n+5}{6} > x$.)
   (b) Show that the set $A_2 = \left\{\frac{n+6}{n} : n \in \mathbb{N}\right\}$ is bounded in the rationals.
5. Show that if $A, B \subseteq \mathbb{Q}$ are both bounded in $\mathbb{Q}$ then $A \cup B \subseteq \mathbb{Q}$ is bounded in $\mathbb{Q}$.
6. Show that for $m \in \mathbb{N}$, $\sqrt{m} \in \mathbb{Q}$ if and only if $\sqrt{m} \in \mathbb{N}$. (Hint: use that if $\sqrt{m} \notin \mathbb{N}$ then we can find $k \in \mathbb{N}$ such that $k < \sqrt{m} < k+1$ and adapt the proof that you saw in lectures that $\sqrt{2} \notin \mathbb{Q}$.)
7. Show that $(\sqrt{3} + \sqrt{5}) \notin \mathbb{Q}$.
8. Let $a, b \in \mathbb{Q}$ with $a < b$ show that there exists an irrational number x such that $a < x < b$.

Analysis 1A exercise sheet 2: Inequalities and induction

1. Prove the following statements (stating which of the axioms for the real numbers that you are using):
   (a) For all $x \in \mathbb{R}$ we have $x \cdot 0 = 0$,
   (b) For all $x, y \in \mathbb{R}$ we have $-(xy) = (-x)y$,
   (c) For all $x, y \in \mathbb{R}$ we have $xy = (-x)(-y)$,
   (d) For all $x, y, z \in \mathbb{R}$ show that $x < y$ and $z < 0$ implies that $xz > yz$.

2. Show that for any real numbers, $a, b > 0$, there exists a natural number, n, such that $na > b$.

3. Show that the set

$$A = \{x^2 + 6x + 6 : x \in \mathbb{R}\}$$

   is bounded below in $\mathbb{R}$. Can you find a greatest lower bound for A?

4. For all $x, y, z \in \mathbb{R}$ prove the following inequalities:
   (a) $||x| - |y|| \leq |x - y|$,
   (b) $|x - y| \leq |x - z| + |z - y|$.

5. (a) Find all $x \in \mathbb{R}$ where $|x - 3| \leq 4$.
   (b) Find all $x \in \mathbb{R}$ where $|x - 2| > 8$.
   (c) Find all $x \in \mathbb{R}$ where $|x - 5| \leq |x - 7|$,
   (d) Find all $x \in \mathbb{R}$ where $x \neq -1$ and $\frac{|x-2|}{|x+1|} \leq 2$.

6. Use the Binomial Theorem to show that for all $x > 0$ and $n \in \mathbb{N}$,

$$(1 + x)^n \geq 1 + nx.$$

7. Prove by induction that for any $x_1, \ldots, x_n \in \mathbb{R}$ we have that

$$\left| \sum_{k=1}^{n} x_k \right| \leq \sum_{k=1}^{n} |x_k|.$$

   (NB: This is Proposition 2.12 in the lecture notes).

8. Prove by induction that for any $n \in \mathbb{N}$ with $n \geq 4$, $n! \geq 2^n$.

9. Show that for any $a, b \in \mathbb{R}$ with $a, b \geq 0$ we have that $\sqrt{ab} \leq \frac{a+b}{2}$.

10. **(Challenging)**. Let $a_1, \ldots, a_n$ be non-negative real numbers. Prove that

$$\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^n \geq \left(\prod_{k=1}^{n} a_k\right)$$

using the following steps:

(a) Prove the result when $a_1 = a_2 = \cdots = a_n$.

(b) Assume that $a_1 \leq a_2 \leq \cdots \leq a_n$ and $a_1 < a_n$. Let $A_n = \frac{1}{n}\sum_{k=1}^{n} a_k$ and show that $a_1 < A_n < a_n$.

(c) Assume that $a_1 \leq a_2 \leq \cdots \leq a_n$ and $a_1 < a_n$. Show that

$$A_n(a_1 + a_n - A_n) - a_1 a_n = (a_1 - A_n)(A_n - a_n) > 0$$

where A_n is as defined above.

(d) Prove the result by induction. Your inductive hypothesis should be that the inequality holds for any n non-negative real numbers. You then need to show this implies the result for non-negative real numbers $0 \leq a_1 \leq \ldots \leq a_{n+1}$ by considering $a_1 + a_{n+1} - A_{n+1}, a_2, \ldots, a_n$.

Analysis 1A exercise sheet 3: Supremum and Infimum

1. Let $x, y \in \mathbb{R}$. Prove the following two statements:
   (a) If for all $\epsilon \in (0, \infty)$ we have that $x - y < \epsilon$ then $x \leq y$.
   (b) If for all $\epsilon \in (0, \infty)$ we have that $|x - y| < \epsilon$ then $x = y$.

2. Let $A, B \subseteq \mathbb{R}$ be non-empty and bounded with $A \subseteq B$. Show that $\sup A \leq \sup B$ and $\inf A \geq \inf B$.

3. Let $A \subseteq \mathbb{R}$ be non-empty and bounded and $B = \{|a| : a \in A\}$. Find $\sup B$ in terms of $\sup A$ and $\inf A$, considering the cases when $|\sup A| \geq |\inf A|$ and when $|\sup A| < |\inf A|$.

4. Find the supremum and infimum of each of the following subsets of $\mathbb{R}$ (justify your answers):
   (a) $A_1 := (1, 2) \cup (2, 3)$,
   (b) $A_2 := \left\{\frac{n-1}{n} : n \in \mathbb{N}\right\}$,
   (c) $A_3 := \left\{\frac{n^2+1}{n-\frac{1}{2}} : n \in \mathbb{Z}\right\}$,
   (d) $A_4 := \left\{\frac{x-1}{x} : x \in (0, \infty)\right\}$,
   (e) $A_5 = \left\{\frac{(-1)^n n+5}{n+1} : n \in \mathbb{N}\right\}$.

5. Let $A, B \subseteq \mathbb{R}$ be non-empty and denote

$$A - B = \{a - b : a \in A \text{ and } b \in B\}.$$

   (a) Show that if $\sup A = \infty$ then $\sup(A - B) = \infty$.
   (b) Show that if A is bounded above and B is bounded below then $\sup(A - B) = \sup A - \inf B$.

6. Let $A \subset (0, \infty)$ be non-empty and $B = \left\{\frac{1}{a} : a \in A\right\}$. Show that if $\inf A = 0$ then $\sup B = \infty$ and that if $\inf A \neq 0$ then $\sup A = \frac{1}{\inf A}$.

7. Let $A = \{x \in \mathbb{R} : x^3 < 2\}$. The aim of this exercise is to show that $\sup A$ is the cube root of 2.
   (a) Show that for all $x, y \in \mathbb{R}$, $x^2 + xy + y^2 \geq 0$ with equality if and only if $x = y = 0$.
   (b) Show that for all $x, y \in \mathbb{R}$ if $x^3 < y^3$ then $x < y$.
   (c) Show that if $y \in \mathbb{R}$ and $y^3 > 2$ then for all $a \in A$, $a < y$.
   (d) Show that if $y \in \mathbb{R}$ and $y^3 > 2$ then $y > \sup A$.
   (e) Show that if $y \in \mathbb{R}$ and $y^3 < 2$ then $y < \sup A$.

(f) Conclude that $(\sup A)^3 = 2$.

8. Determine which of the following subsets of $\mathbb{R}$ have a maximum value:

(a) $A_1 = (0, 1]$,

(b) $A_2 = \left\{\frac{n-1}{n} : n \in \mathbb{N}\right\}$,

(c) $A_3 = \left\{\frac{1}{|n+1/2|} : n \in \mathbb{Z}\right\}$.

Analysis 1A exercise sheet 4: Sequences and limits

1. Determine whether or not the following sequences are bounded (justify your answers):

   (a) $(a_n)_{n\in\mathbb{N}}$ where $a_n = \frac{n+3}{n}$ for all $n \in \mathbb{N}$,

   (b) $(a_n)_{n\in\mathbb{N}}$ where $a_n = \frac{n}{n+2}$ for all $n \in \mathbb{N}$,

   (c) $(a_n)_{n\in\mathbb{N}}$ where $a_n = \frac{n^2+1}{n+3}$ for all $n \in \mathbb{N}$,

   (d) $(a_n)_{n\in\mathbb{N}}$ where $a_n = -n$ for all $n \in \mathbb{N}$.

2. Let $(x_n)_{(n\in\mathbb{N})}$ be the sequence defined by $x_n = \frac{1}{n+4}$ for all $n \in \mathbb{N}$. Show that $\lim_{n\to\infty} x_n = 0$. This should be answered working directly from the definition (that means not using the sandwich rule or Theorem 4.9 from the lecture notes).

3. Let $(x_n)_{n\in\mathbb{N}}$ be the sequence defined by $x_n = \frac{n}{n+2}$ for all $n \in \mathbb{N}$. Show that $\lim_{n\to\infty} x_n = 1$. This should be answered working directly from the definition (that means not using the sandwich rule or Theorem 4.9 from the lecture notes).

4. Show that the following sequences are divergent:

   (a) $(a_n)_{n\in\mathbb{N}}$ where $a_n = 2(-1)^n$ for all $n \in \mathbb{N}$.

   (b) $(a_n)_{n\in\mathbb{N}}$ where $a_n = 6n$ for all $n \in \mathbb{N}$.

   (c) $(a_n)_{n\in\mathbb{N}}$ where $a_n = \frac{n(-1)^n+8}{n}$ for all $n \in \mathbb{N}$.

5. Let $(x_n)_{n\in\mathbb{N}}$ be a convergent real valued sequence with $\lim_{n\to\infty} x_n = x$ for some $x \in \mathbb{R}$. Let $k \in \mathbb{N}$ and (y_n) by the real valued sequence where $y_n = x_{n+k}$ for all $n \in \mathbb{N}$. Prove that $\lim_{n\to\infty} y_n = x$.

6. Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be real valued sequences and $a, b \in \mathbb{R}$. Suppose that $a_n \leq b_n$ for all $n \in \mathbb{N}$, $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Show that $a \leq b$. (Remember to show that $a \leq b$ it suffices to show that for all $\epsilon > 0$, $a - b \leq \epsilon$.)

7. Let $(x_n)_{n\in\mathbb{N}}$ be a real valued sequence and $a \in (0, \infty)$. Show that if $\lim_{k\to\infty} |x_{k+1} - x_k| = a$ then (x_n) is divergent.

8. Let $(x_n)_{(n\in\mathbb{N})}$ be a convergent sequence of non-negative real numbers and suppose that $\lim_{n\to\infty} x_n = x$. Show that $x \geq 0$ and that $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{x}$. **To show that $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{x}$ consider the cases where $x = 0$ and $x > 0$ separately, for the second case it may help to use that**

$$(\sqrt{x_n} - \sqrt{x}) = \frac{(\sqrt{x_n} - \sqrt{x})(\sqrt{x_n} + \sqrt{x})}{\sqrt{x_n} + \sqrt{x}}.$$

9. For each of the following real valued sequences $(a_n)_{n\in\mathbb{N}}$ find $\lim\limits_{n\to\infty} a_n$. Justify your answers, quoting which results from the lectures or earlier questions you are using.

   (a) $a_n = \frac{1}{n^3+5}$ for all $n \in \mathbb{N}$.

   (b) $a_n = \frac{n^2+3}{4n^2+7n}$ for all $n \in \mathbb{N}$.

   (c) $a_n = \sqrt{n+1} - \sqrt{n}$ for all $n \in \mathbb{N}$.

   (d) $a_n = \frac{\sin n + 5n}{n^2}$ for all $n \in \mathbb{N}$. (NB: You may use that $|\sin n| \leq 1$ for all $n \in \mathbb{N}$ without justification.)

10. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers, let $x \in (-\infty, 0)$ and suppose that $\lim\limits_{n\to\infty} x_n = x$. Show that there are only finitely many $k \in \mathbb{N}$ where $x_k \geq 0$.

11. Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be sequences of real numbers. Suppose that (b_n) is bounded and $\lim\limits_{n\to\infty} a_n = 0$. Show that $\lim\limits_{n\to\infty} a_n b_n = 0$.

12. Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be sequences of real numbers and $a \in \mathbb{R}\backslash\{0\}$. Suppose that $(b_n)_{n\in\mathbb{N}}$ is divergent and $\lim\limits_{n\to\infty} a_n = a$. Show that the sequence $(a_n b_n)$ is divergent.

Analysis 1A exercise sheet 5: Sequences divergent to infinity and monotone sequences

1. Show that the following sequences are divergent to $+\infty$:

   (a) $(a_n)_{n\in\mathbb{N}}$ where $a_n = \frac{n^2+1}{n+3}$ for all $n \in \mathbb{N}$.

   (b) $(a_n)_{n\in\mathbb{N}}$ where $a_n = \sqrt{n}$ for all $n \in \mathbb{N}$.

   (c) $(a_n)_{n\in\mathbb{N}}$ where $a_n = (n+1)^2 - n^2$ for all $n \in \mathbb{N}$.

2. (a) Show that if $b \in (0,\infty)$, $n \in \mathbb{N}$ and $n \geq 2$ then $(1+b)^n \geq \frac{n(n-1)b^2}{2}$.

   (b) Show that for all $b \in (0,\infty)$, $\lim\limits_{n\to\infty} (1+b)^n - n = \infty$.

   (c) Show that $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

3. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that $\lim_{n\to\infty} a_n = \infty$. Show that if $c \in (0,\infty)$ then

$$\lim_{n\to\infty} ca_n = \infty \text{ and } \lim_{n\to\infty} -ca_n = -\infty.$$

4. Let $b \in \mathbb{R}$, $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that $\lim\limits_{n\to\infty} a_n = \infty$ and $(b_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that $\lim\limits_{n\to\infty} b_n = b$. Show that

   (a) If $b > 0$ then $\lim\limits_{n\to\infty} b_n a_n = \infty$,

   (b) If $b < 0$ then $\lim\limits_{n\to\infty} b_n a_n = -\infty$,

   (c) For $b = 0$ give examples of sequences $(b_n)_{n\in\mathbb{N}}$ and $(a_n)_{n\in\mathbb{N}}$ where:

      i. $\lim\limits_{n\to\infty} b_n a_n = \infty$,

      ii. $\lim\limits_{n\to\infty} b_n a_n = a$ for some $a \in \mathbb{R}$,

      iii. $\lim\limits_{n\to\infty} b_n a_n = -\infty$.

5. Suppose that (a_n) and (b_n) are sequences where $\lim_{n\to\infty} a_n = \infty$ and there exists $N, k \in \mathbb{N}$ such that $b_{n+k} \geq a_n$ for all $n \in \mathbb{N}$ where $n \geq N$. Show that $\lim_{n\to\infty} b_n = \infty$.

6. Complete the proof of the monotone convergence theorem by showing that if $(a_n)_{n\in\mathbb{N}}$ is monotone decreasing and bounded below then it is convergent. (Rather than try and mimic the case for monotone increasing sequences, consider the sequence with terms $-a_n$.)

7. Let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence and define $(b_n)_{n\in\mathbb{N}}$ by

$$b_n = \inf\{a_k : k \geq n\}.$$

   Show that $(b_n)_{n\in\mathbb{N}}$ is convergent.

8. Show that if $(a_n)_{n\in\mathbb{N}}$ is monotone increasing and unbounded then $\lim_{n\to\infty} a_n = \infty$.

9. Let $(a_n)_{n\in\mathbb{N}}$ be the sequence where $a_1 = 2$ and for all $n \in \mathbb{N}$, $a_{n+1} = \frac{5a_n+2}{2a_n+1}$.

   (a) Show that $0 \leq a_n \leq 1 + \sqrt{2}$ for all $n \in \mathbb{N}$.

   (b) Show that $(a_n)_{n\in\mathbb{N}}$ is convergent.

   (c) Find $\lim\limits_{n\to\infty} a_n$.

10. Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be sequences such that $a_n < b_n$ and

$$(a_{n+1}, b_{n+1}) \subseteq (a_n, b_n)$$

for all $n \in \mathbb{N}$. Show that both (a_n) and (b_n) are convergent.