ANALYSIS 1A EXERCISES: PART 2 of 2

Analysis 1A exercise sheet 6: Subsequences and Cauchy sequences

1. Determine whether the following statements are true or false (justify your answers)

   (a) If (a_n) has a convergent subsequence then (a_n) is convergent.

   (b) If (a_n) has a divergent subsequence then (a_n) is divergent.

   (c) If (a_n) has a bounded subsequence then it has a convergent subsequence.

   (d) If (a_n) has a convergent subsequence it must be bounded.

2. Let $(a_n)_{n\in\mathbb{N}}$ be a real valued sequence.

   (a) Show that if $(|a_n|)_{n\in\mathbb{N}}$ is not divergent to infinity then $(a_n)_{n\in\mathbb{N}}$ contains a convergent subsequence (first show that it has a bounded subsequence).

   (b) Conclude (stating any results from the lecture notes that you use) that $\lim_{n\to\infty}|a_n| = \infty$ if and only if $(a_n)_{n\in\mathbb{N}}$ does not have a convergent subsequence.

3. Determine whether or not the following sequences have convergent subsequences:

   (a) $(a_n)_{n\in\mathbb{N}}$ where $a_n = \frac{n(-1)^n+7}{n+5}$ for all $n \in \mathbb{N}$.

   (b) $(a_n)_{n\in\mathbb{N}}$ where $a_n = \frac{n^2+5}{n+4}$ for all $n \in \mathbb{N}$.

   (c) $(a_n)_{n\in\mathbb{N}}$ where $a_n = (-1)^n n + n$ for all $n \in \mathbb{N}$.

   (d) $(a_n)_{n\in\mathbb{N}}$ where $a_n = (-1)^n n^2 + n$ for all $n \in \mathbb{N}$.

4. Show, directly from the definition, that the following sequences are Cauchy sequences:

   (a) $(a_n)_{n\in\mathbb{N}}$ where $a_n = \frac{3^n+1}{3^n}$ for all $n \in \mathbb{N}$.

   (b) $(a_n)_{n\in\mathbb{N}}$ where $a_n = \frac{5+n}{7n}$ for all $n \in \mathbb{N}$.

   (c) $(a_n)_{n\in\mathbb{N}}$ where $a_1 = 1$ and $a_{n+1} = 1 - a_n/2$ for all $n \in \mathbb{N}$.

5. Show, directly from the definition, that if $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are Cauchy sequences then $(a_n + b_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

6. Determine which of the following statements are true (justify your answers):

(a) Any bounded sequence contains a subsequence which is a Cauchy sequence,

(b) There exists a sequence $(a_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} a_n = \infty$ but where $(a_n)_{n\in\mathbb{N}}$ contains a subsequence which is Cauchy,

(c) Any monotone increasing sequence is a Cauchy sequence.

7. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence where there exists $M > 0$ such that for all $n \in \mathbb{N}$, $\sum_{k=1}^{n} |a_{k+1} - a_k| \leq M$. Show that $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and thus convergent. (Hint: First show that the sequence $(b_n)_{n\in\mathbb{N}}$ defined by $b_n = \sum_{k=1}^{n} |a_{k+1} - a_k|$ for all $n \in \mathbb{N}$ is convergent and thus Cauchy).

8. Let $(a_n)_{n\in\mathbb{N}}$ satisfy that $a_1 = 2$ and $a_{n+1} = 2 + \frac{1}{a_n}$ for all $n \in \mathbb{N}$. Show that:

   (a) For all $n \in \mathbb{N}$, $a_n \geq 2$.

   (b) For all $n \in N$ with $n \geq 2$, $|a_{n+1} - a_n| \leq \frac{1}{4}|a_{n-1} - a_n|$.

   (c) (a_n) is a Cauchy sequence and $\lim_{n\to\infty} a_n = 1 + \sqrt{2}$.

9. Let $(a_n)_{n\in\mathbb{N}}$ satisfy that $a_1 = 1$ and $a_{n+1} = 1 + \frac{1}{a_n}$ for all $n \in \mathbb{N}$.

   (a) Show that if $(a_n)_{n\in\mathbb{N}}$ is convergent and $\lim_{n\to\infty} a_n = \alpha$ then $\alpha^2 = \alpha + 1$ and that we must have $\alpha = \frac{\sqrt{5}+1}{2}$.

   (b) Show that for all $n \in \mathbb{N}$, $a_n \geq 1$ and

   $$|a_{n+1} - \alpha| \leq (\alpha - 1)|a_n - \alpha|$$

   where $\alpha = \frac{\sqrt{5}+1}{2}$. (Remember that this means $1 = \alpha^2 - \alpha$).

   (c) Conclude that $\lim_{n\to\infty} a_n = \frac{\sqrt{5}+1}{2}$.

Analysis 1A exercise sheet 7: Series

1. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence where for all $k \in \mathbb{N}$ either $a_k = 1$ or $a_k = -1$. Let $b_n = \frac{a_n}{2^n}$ for all $n \in \mathbb{N}$. Show that the series $\sum_{n=1}^{\infty} b_n$ is convergent by showing that the sequence of partial sums (S_k), where $S_k = \sum_{n=1}^{k} b_n$, is a Cauchy sequence and thus convergent.

2. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence. Show that if the series $\sum_{n=1}^{\infty} |a_{n+1} - a_n|$ is convergent then the sequence (a_n) is convergent.

3. Show that the series
$$\sum_{n=1}^{\infty} \frac{2}{n(n+2)}$$
is convergent and compute its value.

4. Let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ be sequences of non-negative real numbers. Suppose that the series $\sum_{n=1}^{\infty} a_n$ is convergent and the sequence $(b_n)_{n\in\mathbb{N}}$ is convergent. Show that the series $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

5. Let $n \geq 2$ be a natural number. Let $(a_k)_{k\in\mathbb{N}}$ be a sequence where $a_i \in \{0, \dots n-1\}$ for all $i \in \mathbb{N}$. Show that the series
$$\sum_{k=1}^{\infty} \frac{a_k}{n^k}$$
is convergent.

6. Determine whether the following series are convergent or divergent (justify your answers)
   (a) $\sum_{n=1}^{\infty} \frac{n}{n+6}$,
   (b) $\sum_{n=1}^{\infty} \frac{1}{n^2+3n+4}$,
   (c) $\sum_{n=1}^{\infty} \frac{1}{n^2-6n+3}$,
   (d) $\sum_{n=1}^{\infty} \frac{n}{n^2-6n+3}$.

7. This question asks you to prove the root test. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of non-negative real numbers where the sequence $a_n^{1/n}$ is convergent and $\lim_{n\to\infty} a_n^{1/n} = \lambda$. Show that
   (a) If $\lambda < 1$ then $\sum_{n=1}^{\infty} a_n$ is convergent.
   (b) If $\lambda > 1$ then $\sum_{n=1}^{\infty} a_n$ is divergent.
   (c) If $\lambda = 1$ then it is possible for $\sum_{n=1}^{\infty} a_n$ to be either convergent or divergent. It may help to use the result from Exercise Sheet 5 Question 2(c).

8. Determine whether the following series are convergent or divergent (justify your answers)

   (a) $\sum_{n=1}^{\infty} \frac{6^n}{n!}$,
   (b) $\sum_{n=1}^{\infty} \frac{2^n}{n^6}$,
   (c) $\sum_{n=1}^{\infty} \frac{n}{2^n}$,
   (d) $\sum_{n=1}^{\infty}(\sqrt{n+1} - \sqrt{n})$,
   (e) $\sum_{n=1}^{\infty} \frac{n!+n}{(n+2)!}$.

9. (a) For which values of $x \in (0, \infty)$ is the series $\sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$ convergent.

   (b) For which values of $x \in (0, \infty)$ is the series $\sum_{n=1}^{\infty} \frac{x^n}{2^n}$ convergent.

Analysis 1A exercise sheet 8: Limits and continuity of functions

1. Let $f, g : (0, \infty) \to \mathbb{R}$ be functions and $a, b \in \mathbb{R}$ with $\lim_{x\to\infty} f(x) = a$ and $\lim_{y\to\infty} g(y) = b$.

   (a) Show that if $a > 0$ then there exists $y \in (0, \infty)$ such that if $x > y$ then $f(x) > 0$.

   (b) Show that if $f(x) < g(x)$ for all $x \in (0, \infty)$ then $a \leq b$.

2. Let $f : (0, \infty) \to \mathbb{R}$ be defined by $f(x) = \frac{x-3}{x}$. Show that $\lim_{x\to\infty} f(x) = 1$ and that $\lim_{x\to 0} f(x) = -\infty$.

3. Let $A \subset \mathbb{R}$, $f, g : A \to \mathbb{R}$ be functions and a be an accumulation point of A. Let $b, c \in \mathbb{R}$ and suppose that $\lim_{x\to a} f(x) = b$ and $\lim_{x\to a} g(x) = c$. Show that:

   (a) $\lim_{x\to a}(f + g)(x) = b + c$

   (b) $\lim_{x\to a} f \cdot g(x) = bc$

   (c) If $g(x) \neq 0$ for all $x \in A$ and $c \neq 0$ then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{b}{c}.$$

4. Let $A \subset \mathbb{R}$, $f : A \to \mathbb{R}$ be a function and a be an accumulation point of A. Show that if $b \in \mathbb{R}$ and $\lim_{x\to a} f(x) = b$ then $\lim_{x\to a} |f(x)| = |b|$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a function which satisfies that if $x, y \in \mathbb{R}$ and $x \leq y$ then $f(x) \leq f(y)$ (f is increasing). Show that

   (a) If f is bounded above then

$$\lim_{x\to\infty} f(x) = \sup\{f(x) : x \in \mathbb{R}\}.$$

   (b) If f is unbounded above then

$$\lim_{x\to\infty} f(x) = \infty.$$

6. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that for all $x \in \mathbb{R}$, $|f(x)| \leq |x|$. Show that $f(0) = 0$ and $\lim_{x\to 0} f(x) = 0$ (in other words show that f is continuous at 0).

   (b) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ where f is continuous at 0 but discontinuous at all $x \in \mathbb{R}$ with $x \neq 0$.

7. Let $f, g : [0, \infty] \to \mathbb{R}$ be functions which are continuous at 0. Show that if $f(0) - g(0) = 1$ then there exists $\delta \in (0, \infty)$ such that if $x \in [0, \delta]$ then $f(x) - g(x) \geq \frac{1}{2}$.

8. Let $A \subset \mathbb{R}$, $a \in A$ and $f, g : A \to \mathbb{R}$ be functions which are continuous at a.

   (a) Show that $f + g : A \to \mathbb{R}$ is continuous at a.

   (b) Show that $|f| : A \to \mathbb{R}$ is continuous at a.

   (c) Show that $f \cdot g : A \to \mathbb{R}$ is continuous at a.

   (d) Show that if $g(x) \neq 0$ for any $x \in A$ and we define a function $h : A \to \mathbb{R}$ by $h(x) = f(x)/g(x)$ for all $x \in A$ then h is continuous at a.

9. Define $f : [0, \infty) \to \mathbb{R}$ by $f(x) = \sqrt{x}$. Show that f is continuous on $[0, \infty)$.

10. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \sup\{k \in \mathbb{Z} : k \leq x\}$ (this function is often called the integer part of x and denoted $\lfloor x \rfloor$). For which elements of $\mathbb{R}$ is f continuous and for which elements in $\mathbb{R}$ is f discontinuous.

Analysis 1A exercise sheet 9: The extremal value theorem and the intermediate value theorem

1. Let $f : [0, 1] \to \mathbb{R}$ be continuous. Show that if $(x_n)_{n\in\mathbb{N}}$ is a sequence where $x_n \in [0, 1]$ for all $n \in \mathbb{N}$ then the sequence $(f(x_n))_{n\in\mathbb{N}}$ has a convergent subsequence.

2. Let $f : (0, 1) \to \mathbb{R}$ be continuous on $(0, 1)$ with
$$\lim_{x\to 0} f(x) = \lim_{x\to 1} f(x) = 0$$
and where $f(x) > 0$ for all $x \in (0, 1)$. Show that:
   (a) there exists $z \in (0, 1)$ such that $f(z) = \sup\{f(x) : x \in (0, 1)\}$,
   (b) there does not exist $z \in (0, 1)$ such that $f(z) = \inf\{f(x) : x \in (0, 1)\}$.

3. Let $f : [0, \infty) \to \mathbb{R}$ be continuous on $[0, \infty)$ with $f(0) = 1$ and $\lim_{x\to\infty} f(x) = 0$. Show that there exists $z \in [0, \infty)$ such that
$$f(z) = \sup\{f(x) : x \in (0, \infty)\}.$$
Is it always true that there will exist $z \in [0, \infty)$ such that
$$f(z) = \inf\{f(x) : x \in (0, \infty)\}?$$

4. (a) Show that there exists $x \in (0, 1)$ such that $x^4 + x^3 + 3x - 4 = 0$.
   (b) Show that the equation $2x^4 + x^3 + 3x^2 - 2 = 0$ has at least two solutions in $(-1, 1)$.
   (c) Show that there exists $x \in (0, 1)$ such that $x^5 + 1 = 4x$.

5. Let $f : [0, 1] \to \mathbb{R}$ be continuous on $[0, 1]$ and satisfy that $f(x) > 0$ for all $x \in [0, 1]$. Show there exists $a > 0$ such that $f(x) \geq a$ for all $x \in [0, 1]$.

6. Let $f : [0, 1] \to [0, 3]$ be continuous on $[0, 1]$. Show that there exists $x \in [0, 1]$ such that $f(x) = 3x$.

7. Let $f : [0, \infty) \to \mathbb{R}$ be continuous where $f(0) = 1$ and $\lim_{x\to\infty} f(x) = 0$. Show that there exists $x \in (0, \infty)$ where $f(x) = x$.

8. Let $f : [0, 1] \to \mathbb{R}$ be continuous. Show that the range of f,
$$f([0, 1]) = \{f(x) : x \in [0, 1]\} = [a, b]$$
for some $a, b \in \mathbb{R}$ with $a \leq b$. First use the extremal value theorem and then use the intermediate value theorem.

9. Let $f : [0, 1] \to \mathbb{R}$ be continuous and suppose that there exist $x, y \in [0, 1]$ with $f(x)f(y) < 0$. Show that there exists $z \in [0, 1]$ such that $f(z) = 0$.