ANALYSIS 1A

Notes by Dr Thomas Jordan – Part 1 of 2

Unit Administration

Teaching methods

There will be two lectures (Tuesday 5pm and Friday 2pm) and one problems class (Friday 3pm) each week, all of which take place in the Tyndall Lecture Theatre. The lectures will introduce new material and the problems class will go over examples of problems and how to start your homework. There will be a weekly homework set which you will hand in to your tutor, who you will see for weekly tutorials.

Assessment

In addition to the weekly homeworks there will be two assessed homeworks. These homeworks will be issued in weeks 4 and 8 and will in total count for 10 percent of your mark in this unit. The remaining 90 percent of the unit mark will come from the January exam.

Website

Notes, problems, solutions to problems, sample exam, and further information will appear on the website
`https://people.maths.bris.ac.uk/~malhw/`

Textbooks

Both the books

1. Elementary Analysis by Kenneth A. Ross
2. Real Analysis by John M. Howie

cover the material in this unit (and much more).

Work expected

You are expected to spend 3-4 hours per week working on this course outside of the lectures and tutorials. In this time you are expected to review the lecture notes and the comments on your homework, and to work on your current homework assignment. It is usually best to first do rough work for your homework and then to rewrite your homework solutions clearly before submitting them to your tutor. Your tutor will arrange with you the due dates of your weekly homework assignments; the due dates of the assessed homework assignments will be announced soon before they are distributed.

List of sections and topics

1. Introdution: motivation to study analysis
2. The rational and real numbers (the integers and the rationals; the Completeness axiom; real numbers; bounds for sets; inequalities; induction; the Binomial theorem)
3. Supremum and infimum (definitions; calculating suprema and infima)
4. Sequences (definition and basic properties of sequences; definition and basic properties of convergent sequences; definition of limits of sequences; arithmetic properties of limits of sequences; divergent sequences)
5. Monotonic sequences subsequences and Cauchy sequences (definition of monotonic sequences; the Monotone convergence theorem; computing limits of monotonic sequences; defintion and examples of subsequences; the Bolzano-Weierstrass theorem; definition of Cauchy sequences; Cauchy's principle of convergence)
6. Series (definition and theorems on computing series; the Ratio test; the Root test)
7. Functions (bounded functions; accumulation points; limits of functions; the Heine definition of the limit of a function; continuity of a function; Extremal value theorem; Intermediate value theorem)

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1 Introduction: motivation to study analysis

Before taking this unit you will probably already have looked at differentiating and integrating functions. However in the definition of the derivative of a function f at a point x you will have probably seen

$$\lim_{y \to x} \frac{f(y) - f(x)}{y - x}.$$

For a lot of functions it is relatively clear what this limit should be but you may well not have seen a formal definition of the limit of a function at a point. In fact it is not hard to construct examples where talking about this limit does not make sense.

Example. *Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by*

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } \quad x \leq 0 \end{cases}$$

and called the absolute value of x. If we try and differentiate this function at $x = 0$ we hit difficulties. If $y > 0$ then we have that $\frac{f(y)-f(0)}{y} = 1$ but if $y < 0$ then $\frac{f(y)-f(0)}{y} = -1$. So it does not make sense to talk about $\lim_{y \to 0} \frac{f(y)-0}{y}$ and we cannot differentiate this function at 0. However it is easy to differentiate it anywhere else and also to integrate.

Example. *Consider the function*

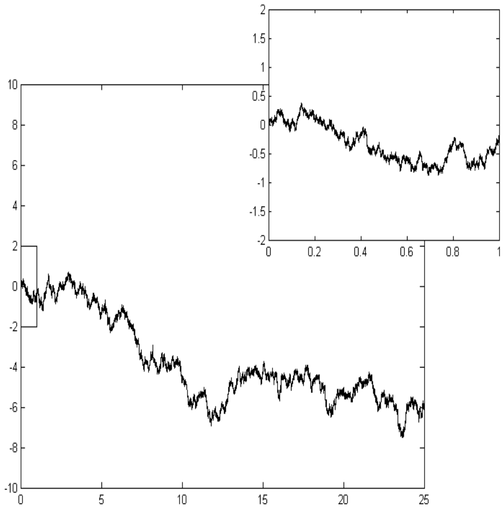
$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

Again if we try and differentiate this function we will hit problems since

$$\frac{x \sin(1/x) - 0}{x} = \sin\left(\frac{1}{x}\right)$$

and this has no limit as x goes to 0.

Example. *Brownian Motion is a random process which can be used to model behaviour in several different areas including Biology, finance and physics. In one dimensional Brownian motion is a random process which can be described in the following way $B(0) = 0$, $B(t + h)$ is normally distributed (bell shaped curve) with expectation t and variance h. In fact it is hard to see whether this definition makes sense (to see it introduced properly you'll need to take the higher level probability units.) However it turns out that a typical graph of Brownian motion looks like*



[1] *and is typically differentiable nowhere. So to study this type of process you need to go beyond the calculus where only smooth functions are considered.*

In order to be able to deal with this type of situation it is necessary to have a clear understanding of what a limit is. Analysis is essentially the study of limits which in turn will lead to studying continuity (Analysis 1A), differentiation (Analysis 1B) and integration (Analysis 1B).

[1] 'Wiener process zoom'. Licensed under Creative Commons Attribution-Share Alike 3.0 via Wikimedia Commons, `http://commons.wikimedia.org/wiki/File:Wiener_process_zoom.png#mediaviewer/File:Wiener_process_zoom.png`

2 The rational and real numbers

2.1 The integers and the rationals

We will use notation from set theory throughout these notes. We will simply think of a set as a finite or infinite collection of objects (usually the objects will be numbers but not always). To denote a set we will list the elements of the set in curly brackets (e.g. $\{1, 2, 3\}$ would be the set containing 1, 2 and 3).

Notation. *For a sets A, B we will use the following notation*

1. *$a \in A$ means that a is an element of A. $a \notin A$ means that a is not an element of A. For example $1 \in \mathbb{N}$ but $\frac{1}{2} \notin \mathbb{N}$.*
2. *$A \subseteq B$ means that A is a subset of B. This means that any element of A is an element of B. For example $\mathbb{Z} \subseteq \mathbb{Q}$.*
3. *$A \cup B = \{a : a \in A \text{ or } a \in B\}$ denotes the union of the sets A and B. Whereas $A \cap B = \{a : a \in A \text{ and } a \in B\}$.*
4. *$\emptyset$ denotes the empty set which is the set containing no elements.*
5. *For sets A and B we let*
$$A \backslash B = \{x \in A : x \notin B\}.$$

The natural numbers are

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

(note that this is not consistent throughout mathematics you will sometimes see 0 listed as a natural number) and come with two natural operations addition, $+$ and multiplication, $\cdot$ as well as a natural ordering. We can extend this to the integers

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

and to the rational numbers

$$\mathbb{Q} = \left\{ \frac{p}{q} : p \in \mathbb{Z} \text{ and } q \in \mathbb{N} \right\}.$$

Note it is often convenient to write rational numbers in the form $\frac{p}{q}$ where p and q have no common divisors in which case we write $(p, q) = 1$ (for more details see Foundations and Proof). We can also extend addition, multiplication and the ordering from the natural numbers to the rational numbers. For rational numbers $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$ we have

$$\frac{p_1}{q_1} + \frac{p_2}{q_2} = \frac{p_1 q_2 + p_2 q_1}{q_1 q_2} \text{ and } \frac{p_1}{q_1} \cdot \frac{p_2}{q_2} = \frac{p_1 p_2}{q_1 q_2}$$

and

$$\frac{p_1}{q_1} < \frac{p_2}{q_2} \text{ if and only if } p_1 q_2 < q_1 p_2.$$

In fact we can easily show that the following arithmetic and ordering properties are satisfied

(A1). Closure under addition. For all $x, y \in \mathbb{Q}$ we have that $x + y \in \mathbb{Q}$.

(A2). Associativity under addition. For all $x, y, z \in \mathbb{Q}$ we have that $x + (y + z) = (x + y) + z$.

(A3). Zero is the additive identity. For all $x \in \mathbb{Q}$ we have $x + 0 = x$.

(A4). Every element has an additive inverse. For all $x \in Q$ we have $-x \in \mathbb{Q}$ and $x + (-x) = 0$.

(A5). Addition is commutative. For all $x, y \in \mathbb{Q}$ we have $x + y = y + x$.

(A6). Closure under multiplication. For all $x, y \in \mathbb{Q}$ we have $xy \in \mathbb{Q}$.

(A7). Associativity under multiplication. For all $x, y, z \in \mathbb{Q}$ we have $(xy)z = x(yz)$.

(A8). One is the multiplicative identity. For all $x \in \mathbb{Q}$ we have $1 \cdot x = x$.

(A9). All nonzero rationals have a multiplicative inverse. For all $x \in \mathbb{Q}$ where $x \neq 0$ we have that $x^{-1} = \frac{1}{x} \in \mathbb{Q}$ and $x^{-1}x = 1$.

(A10). Multiplication is commutative. For all $x, y \in \mathbb{Q}$ we have that $xy = yx$.

(A11). Distributivity law. For all $x, y, z \in \mathbb{Q}$ we have $x(y + z) = xy + xz$.

(O1). Transitivity. For all $x, y, z \in \mathbb{Q}$ if $x < y$ and $y < z$ then $x < z$.

(O2). Trichotomy. For all $x, y \in \mathbb{Q}$ either $x < y$, $x = y$ or $y < x$.

(O3). Compatibility with addition. For all $x, y, z \in \mathbb{Q}$ if $x < y$ then $x + z < y + z$.

(O4). Compatibility with multiplication. For all $x, y, z \in Q$ if $x < y$ and $z > 0$ then $zx < zy$.

A formal way of saying this would be that the rationals are an ordered field under addition and multiplication (you will see more about this in linear algebra). In fact any set satisfying these properties must contain the rational numbers. The rational numbers are easy to define from the natural numbers however it will turn out they are not sufficient for our needs.

2.2 Bounds for sets and irrationals

A couple of notions we will want to use extensively throughout this course is the absolute value and the notion of a subset being bounded.

Definition 2.1. *For $x \in \mathbb{Q}$ we will denote the absolute value of x by*

$$|x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x \leq 0 \end{cases}$$

A useful way of thinking of the absolute value $|x|$ is the distance of x from 0. (So $|x - y|$ is the distance from x to y.)

Definition 2.2. *We say a set $A \subseteq \mathbb{Q}$ is bounded above by $\alpha \in \mathbb{Q}$ if for all $x \in A$ we have $x \leq \alpha$. Similarly we say that a set is bounded below by $\alpha \in \mathbb{Q}$ if for all $x \in A$ we have that $x \geq \alpha$. If A is bounded above and below we say that A is bounded (alternatively A is bounded if there exists $\alpha \in \mathbb{Q}$ such that for every $x \in A$ we have $|x| \leq \alpha$.)*

If we consider a subset A of integers which is bounded above then A will contain an upper bound for A and this will be the least upper bound for A. Similarly a subset of integers of A which is bounded below will contain a lower bound for A which will be the greatest lower bound for A. However if we work with any bounded subset of the rational numbers we will not always be able to find a least upper bound without going beyond the rational numbers. We will see an example of this using the fact that 2 cannot have a rational square root.

Notation. *For a non-negative number, x we will let $\sqrt{x}$ denote the non-negative square root of x, that is $\sqrt{x} = y$ if $y \geq 0$ and $y^2 = x$.*

It is not always true that the square root of a non-negative rational number is still a rational number.

Theorem 2.3. *$\sqrt{2}$ is not rational.*

Proof. We will give a proof which is slightly different to the usual proof in that it does not use divisibility. We use a proof by contradiction. This means we assume that the statement we want to proof is not true and show that this leads to a contradiction. So we assume that $\sqrt{2} \in \mathbb{Q}$ in which case we can find a positive integer r such that $\sqrt{2}r \in \mathbb{Z}$. We now take q to be the smallest positive integer such that $\sqrt{2}q \in \mathbb{Z}$ (note that since the set of positive integers are a set of integers and bounded below such an element exists). However we then have that

$$q(\sqrt{2} - 1) = q\sqrt{2} - q \in \mathbb{Z}$$

and since $0 < \sqrt{2} - 1 < 1$ we have that $0 < q(\sqrt{2} - 1) < q$. However

$$\sqrt{2}(q(\sqrt{2} - 1)) = 2q - q\sqrt{2} \in \mathbb{Z}$$

and so $\sqrt{2}(q(\sqrt{2}-1)) \in \mathbb{Z}$ but since $q(\sqrt{2}-1)$ is a positive integer less than q this contradicts the definition of q. □

This means that if we want to just use the rational numbers then there would be simple geometric problems we could not answer.

Example. *What is the distance of the two points on the plane* $(1, 0)$ *and* $(0, 1)$*. By Pythagoras's Theorem the answer is* $\sqrt{2}$*.*

It also means that the rational numbers do not have the property that every set which is bounded above has a least upper bound in the rationals.

Example. *Consider the set*

$$A = \{x \in \mathbb{Q} : x^2 < 2\}.$$

We can see that if $x \geq 2$ *then* $x \notin A$ *so* A *is bounded above. However* A *does not have a least upper bound in the rationals (why?).*

The real numbers will be defined to be an extension of the rational numbers where every set bounded above has a least upper bound. We will define the real numbers $\mathbb{R}$ to satisfy all of properties (A1) to (A11) and (O1) to (O4) and in addition the following axiom.

Definition 2.4 (Completeness axiom)**.** *Every non-empty subset* A *of* $\mathbb{R}$ *which is bounded above must have a least upper bound.*

It turns out that this uniquely defines the real numbers (we do not have time to justify this here). There are ways of constructing the real numbers from the rational numbers, if you are interested in how this is done look up Dedekind cuts or the Stevin construction. The simplest way of thinking of real numbers is by the Stevin construction which is to represent them by infinite decimal expansions, however to show that this satisfies the axioms above is non-trivial.

Definition 2.5 (Real numbers)**.** *We define the real numbers* $\mathbb{R}$ *to satisfy all of axioms (A1) to (A11) and (O1) to (O4) as well as the completeness axiom.*

Remark. *The notions of when a subset is bounded and the absolute value are the same as for the rational numbers.*

There are some important properties of the real numbers we can deduce immediately from these axioms.

Proposition 2.6. *Every non-empty subset of real numbers bounded below has a greatest lowest bound.*

Proof. Let $A \subseteq \mathbb{R}$ be bounded below with lower bound C. Consider the set

$$B = \{-x : x \in A\}.$$

For any $x \in B$ we have that $-x \in A$ and so $-x \geq C$. We can then use the order axioms to show that $x \leq -C$ (**how?**). Thus B is bounded above and non-empty and so by the completeness axiom has a least upper bound U. If we let $L = -U$ then for all $x \in A$ we have that $x \geq U$ (**why?**). Now since U is the least upper bound for the set B we know that if we take $y > L$ then $-y < U$ and $-y$ is not an upper bound for B so we can find $b \in B$ such that $b > -y$. However this means that $-b \in A$ and $-b < y$ so y cannot be a lower bound for A and so L is the greatest lower bound for A. □

Remark. *This proof illustrates in a simple setting what you need to do to show a number α is the greatest lower bound for a set A. You need to show that for all $x \in A$ we have that $x \geq \alpha$ and for any $\beta > \alpha$ we can find $x \in A$ such that $x < \beta$ (the analogous argument works for least upper bounds). More on this later.*

Proposition 2.7 (Archimedean property)**.** *For any $x \in \mathbb{R}$ there exists $k \in \mathbb{N}$ such that $k \geq x$.*

Proof. Fix $x \in \mathbb{R}$ and suppose that no such integer exists. This means that for all $z \in \mathbb{N}$ we have that $z < x$ and so the natural numbers are bounded above by x. Thus by the completeness axiom the integers must have a least upper bound α. Since α is a least upper bound this must mean that $\alpha - \frac{1}{2}$ is not an upper bound for $\mathbb{N}$. So we can conclude that there exist a natural number z between $\alpha - \frac{1}{2}$ and α. However in this case $z + 1$ is an integer greater than α and we have a contradiction. □

Finally we will justify that for non-negative real numbers it makes sense to talk about the square root.

Proposition 2.8. *For any $x \in \mathbb{R}$ with $x \geq 0$ there exists $y \in \mathbb{R}$ with $y \geq 0$ and $y^2 = (-y)^2 = x$. So for any $y \in \mathbb{R}$ with $x \geq 0$ we have that $\sqrt{y} \in \mathbb{R}$.*

Proof. If $x = 0$ we can simply take $y = 0$ so we can assume that $x > 0$. We let $x > 0$ and

$$A = \{z \in \mathbb{R} : z^2 < x\}.$$

This set is clearly non-empty as it contains 0. It is also bounded as by the Archimedean principle we can find a natural number $n > x$ and for any $a \geq n$ we will have that

$$a^2 \geq n^2 \geq n > x.$$

So we can conclude that $a \notin A$ and n is an upper bound for the set A. Thus by the completeness axiom we can find a least upper bound for the set A

which we will denote by y. We will let $y^2 = \alpha$. Our aim is to show that $\alpha = x$. We let $1 > \epsilon > 0$, use the Archimedean principle to choose $n \in \mathbb{N}$ with $n \geq y$ and note that

$$(y + \epsilon)^2 = y^2 + 2y\epsilon + \epsilon^2 \leq \alpha + \epsilon(1 + 2y) \leq \alpha + \epsilon(1 + 2n).$$

Now we know that since y is an upper bound for A which means $(y+\epsilon) \notin A$ and so $(y + \epsilon)^2 > x$. So $\alpha + (1 + 2n)\epsilon > x$ and since we can take ϵ to be arbitrarily small this means $\alpha \geq x$ (**think about how to justify this argument, it crops up all the time in analysis**).

We now know that $y^2 = \alpha \geq x > 0$ and since $0 \in A$ we have that $y \geq 0$. Thus we can conclude that $y > 0$. Now we consider $(y - \epsilon)^2$ for $y > \epsilon > 0$. Since y is the least upper bound for A we can deduce that $(y - \epsilon)^2 \leq x$ (**why?**). We then have that

$$x \geq (y - \epsilon)^2 = y^2 - 2\epsilon y + \epsilon^2 \geq \alpha - 2n\epsilon.$$

So we have that for all $\epsilon > 0$, $x + 2n\epsilon \geq \alpha$ and so we can conclude that $x \geq \alpha$. So we have shown that $x \leq \alpha$ and $x \geq \alpha$ so we can conclude that $x = \alpha$. □

Remark. *This proof has made use of some important techniques in analysis. One is that for $a, b \in \mathbb{R}$ to show that $a = b$ we can first show that $a \leq b$ and then show $b \leq a$. Secondly for $a, b \in \mathbb{R}$ to show that $a \leq b$ it is sufficient to show that for all $\epsilon > 0$, $a < b + \epsilon$ (why is this true?)*

2.3 Inequalities and induction

The study of analysis is full of inequalities. In this part of the notes we prove some important inequalities some of which will be used repeatedly throughout the course. Hopefully it should also help familiarise (or remind) you with some techniques for working with inequalities. Several of these inequalities involve the absolute value so we start with a straightforward result on absolute values.

Proposition 2.9. *For all $x \in \mathbb{R}$ we have that*

1. $|x| \geq 0$ *with* $|x| = 0$ *if and only if* $x = 0$.
2. $|xy| = |x||y|$
3. $|x^2| = |x|^2 = x^2$

The next result is an inequality which will be used extensively throughout the unit.

Proposition 2.10 (Triangle inequality)**.** *For all $x, y \in \mathbb{R}$ we have*

$$|x + y| \leq |x| + |y|.$$

Proof. We proof this by splitting it into two cases and use the simple fact that for all $x \in \mathbb{R}$, $x \leq |x|$ and $-x \leq |x|$. Let $x, y \in \mathbb{R}$
Case 1: If $x \geq -y$ then $x + y \geq 0$ and

$$|x + y| = x + y \leq |x| + |y|.$$

Case 2: If $x < -y$ then $x + y < 0$ and

$$|x + y| = -(x + y) = -x - y \leq |x| + |y|.$$

□

Remark. *We can use the absolute value to define a distance between any two points $x, y \in \mathbb{R}$ by $d(x, y) = |x - y|$. It can be deduced from the results above that for all $x, y, z \in \mathbb{R}$ that $d(x, y) \geq 0$ with $d(x, y) = 0$ if and only if $x = y$ and $d(x, y) \leq d(x, z) + d(z, y)$. These properties show it makes sense to call d a distance.*

The following result gives some very basic inequalities which are helpful to use when proving other inequalities.

Proposition 2.11. *For $a, b, c \in \mathbb{R}$ we have that*

1. $a^2 \geq 0$
2. *If $a, b \geq 0$ then $a \geq b$ if and only if $a^2 \geq b^2$.*
3. *If $a > 0$ then $4ac - b^2 \geq 0$ if and only if for all $x \in \mathbb{R}$, $ax^2 + bx + c \geq 0$.*

Proof. The first two parts are easy exercises. For the third part we complete the square (this should be familiar for the procedure to solve quadratic equations). Suppose that $a > 0$ and $4ac - b^2 \geq 0$ and $x \in \mathbb{R}$ then

$$\begin{aligned} x^2 + \frac{bx}{a} + \frac{c}{a} &= \left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2} \\ &= \left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}. \end{aligned}$$

We know that both $\left(x + \frac{b}{2a}\right)^2$ and $\frac{4ac-b^2}{4a^2}$ are positive and so $x^2 + \frac{bx}{a} + \frac{c}{a} \geq 0$. Since $a > 0$ we can conclude that $ax^2 + bx + c \geq 0$.

On the other hand suppose that $a > 0$ and for all $x \in \mathbb{R}$ we have that $ax^2 + bx + c \geq 0$. This means that $0 \leq \left(x + \frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}$ for all $x \in \mathbb{R}$ and in particular for $x = \frac{-b}{2a}$. Substituting this in gives that $0 \leq \frac{4ac-b^2}{4a^2}$ and thus $4ac - b^2 \geq 0$.

□

We now give examples of using the above three results:

Examples. *We will first show that for all* $a, b \in \mathbb{R}$ *we have that* $a^2 + b^2 \geq 2ab$. *Let* $a, b \in \mathbb{R}$. *From part 1 of Proposition 2.11 we know that* $(a-b)^2 \geq 0$ *and so* $a^2 + b^2 - 2ab \geq 0$ *which means that* $a^2 + b^2 \geq 2ab$.

To illustrate part 2 of Proposition 2.11 we will prove the triangle inequality without doing a case by case argument. Let $x, y \in \mathbb{R}$ *and note that* $|x+y|, |x| + |y| \geq 0$ *so to show* $|x+y| \leq |x| + |y|$ *it suffices to show that* $|x+y|^2 \leq (|x| + |y|)^2$. *To do this we have*

$$|x+y|^2 = (x+y)^2 = x^2 + y^2 + 2xy \leq x^2 + y^2 + 2|x||y| = (|x| + |y|)^2.$$

Finally to illustrate part 3 of Proposition 2.11 we will prove the Cauchy-Schwarz inequality. This says that for $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$ *we have that*

$$(a_1 b_1 + \cdots + a_n b_n)^2 \leq (a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2).$$

To prove this we let $\lambda \in \mathbb{R}$ *and note that for each* $1 \leq i \leq n$, $(\lambda a_i - b_i)^2 \geq 0$ *and so*

$$\sum_{i=1}^{n} (\lambda a_i + b_i)^2 \geq 0$$

which we can multiply out to get

$$\lambda^2 \left(\sum_{i=1}^{n} a_i^2 \right) + \left(\sum_{i=1}^{n} 2a_i b_i \right) \lambda + \sum_{i=1}^{n} b_i^2 \geq 0$$

This holds for all $\lambda \in \mathbb{R}$. *Note that if* $a_1 = a_2 = \cdots = a_n = 0$ *then the inequality is easy so we may assume that* $\sum_{i=1}^{\infty} a_i^2 > 0$. *So by part 3 of Proposition 2.11 we must have that* $4ac - b^2 \geq 0$ *where* $a = \sum_{i=1}^{n} a_i^2$, $b = (\sum_{i=1}^{n} 2a_i b_i)$ *and* $c = \sum_{i=1}^{n} b_i^2$. *Thus*

$$4 \left(\sum_{i=1}^{n} a_i^2 \right) \left(\sum_{i=1}^{n} b_i^2 \right) - \left(\sum_{i=1}^{n} 2a_i b_i \right)^2 \geq 0$$

and we rearrange to get that

$$(a_1 b_1 + \cdots + a_n b_n)^2 \leq (a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2).$$

When showing that $\sqrt{2}$ is irrational we used the method of proof by contradiction. This is when you assume the statement you want to prove is false and then show that this assumption implies a false statement (**Note that assuming the statement is true and deducing a true statement does not prove the statement is true!** Another important method of proof is proof by induction. The idea here is that if we want to prove the statement $P(n)$ is true for all $n \in \mathbb{N}$ it is sufficient to prove that $P(1)$ is true and that for all $k \in \mathbb{N}$ we have $P(k) \implies P(k+1)$. We'll finish the section by giving some examples of this method.

Proposition 2.12. *For all $n \in \mathbb{N}$ we have that $2^n > n$.*

Proof. So for each $n \in \mathbb{N}$ $P(n)$ is the statement that $2^n > n$. So $P(1)$ says that $2 > 1$ and is true. We now fix an arbitrary $k \in \mathbb{N}$ and assume that $P(k)$ is true (we want to show that this means $P(k+1)$ must also be true). So we are assuming $2^k > k$ we have

$$2^{k+1} = 2 \cdot 2^k > 2k = k + k \geq k + 1.$$

and so $P(k)$ implies $P(k+1)$ and we have proved the result by induction. □

Induction can be used to extend the triangle inequality as follows.

Proposition 2.13. *For all $n \in \mathbb{N}$, for all $x_1, \ldots, x_n \in \mathbb{R}$ we have that*

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|.$$

Proof. Exercise □

Finally we complete this section by using induction to prove the Binomial Theorem. To start we need to define the binomial coefficients

Definition 2.14. *For $n \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{0\}$ with $k \leq n$ we define*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n \cdots (n-k+1)}{k!}$$

where $n! = n(n-1)(n-2)\cdots 1$ and $0! = 1$.

We will need the following simple identity

Lemma 2.15 (Pascal's triangle identity)**.** *For $n, k \in \mathbb{N}$ with $2 \leq k \leq n$ we have that*

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}.$$

Proof. Fix $n, k \in \mathbb{N}$ with $2 \leq k \leq n$. We can then rearrange to get

$$\begin{aligned}
\binom{n}{k-1} + \binom{n}{k} &= \frac{n \cdots (n-k+2)}{(k-1)!} + \frac{n \cdots ((n-k+1)}{k!} \\
&= \frac{kn \cdots (n-k+2) + n \cdots (n-k+1)}{k!} \\
&= \frac{n \cdots (n-k+2)}{k!}(k + (n-k+1)) \\
&= \frac{(n+1)n \cdots (n-k+2)}{k!} \\
&= \frac{(n+1)n \cdots ((n+1)-k+1)}{k!} = \binom{n+1}{k}.
\end{aligned}$$

□

Remark. *The reason the above result is called the Pascal's triangle identity is that it shows that Pascal's triangle can be written using binomial coefficients. See* `http://www.mathsisfun.com/pascals-triangle.html`

Theorem 2.16 (Binomial Theorem). *For all $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ we have that*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Proof. We will fix $x, y \in \mathbb{R}$ and for $n \in \mathbb{N}$ let $P(n)$ be the statement that

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^j y^{n-j}.$$

We will prove $P(n)$ is true by induction. $P(1)$ is the statement that $(x+y)^1 = x+y$ and so is clearly true. We will assume that $P(k)$ is true for some $k \in \mathbb{N}$. That is

$$(x+y)^k = \sum_{j=0}^{k} \binom{k}{j} x^j y^{k-j}.$$

We then have that

$$(x+y)^{k+1} = (x+y) \left(\sum_{j=0}^{k} \binom{k}{j} x^j y^{k-j} \right). \tag{1}$$

We need to expand the right hand side of (1). This will give us a summations of terms of the form $x^i y^{(k+1)-i}$ where $0 \le i \le k+1$ multiplied by coefficients. We can see that the coefficients for x^{k+1} $(i = k+1)$ and y^{k+1} $(i = 0)$ are both 1 so we want to find the coefficients for the terms $x^i y^{(k+1)-i}$ where $1 \le i \le k$. In this case the coefficient for $x^i y^{(k+1)-i}$ will be the coefficient on the right side of (1) for $x^i y^{k-i}$ added to the coefficient for $x^{i-1} y^{k-(i-1)}$. By Lemma 2.15 this gives

$$\binom{k}{i} + \binom{k}{i-1} = \binom{k+1}{i}.$$

So we have that

$$(x+y)^{k+1} = \sum_{j=0}^{k+1} \binom{k+1}{j} x^j y^{k+1-j}.$$

So we have shown that $P(1)$ is true and that for all $k \in \mathbb{N}$, $P(k)$ implies $P(k+1)$. Thus by induction $P(n)$ is true for all $n \in \mathbb{N}$. □

3 Supremum and Infimum

Before starting with supremum and infimum we give one important remark and an important definition. The first of which is a simple remark about how to show two real numbers are equal.

Remark. *If $x, y \in \mathbb{R}$ and we want to show that $x = y$ it is not always easy to do this by direct calculations. Thus the following two simple results give useful methods of doing this*

1. *If $x, y \in \mathbb{R}$, $x \leq y$ and $y \leq x$ then $x = y$.*
2. *If $x, y \in \mathbb{R}$ and for all $\epsilon \in \mathbb{R}$ with $\epsilon > 0$ we have that $|x - y| < \epsilon$ then $x = y$. (See exercise sheet 3).*

We also want to introduce some important subsets of the real numbers.

Definition 3.1. *We let $a, b \in \mathbb{R}$ with $a \leq b$. We define:*

1. $(a, b) = \{x \in \mathbb{R} : a < x < b\}$,
2. $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$,
3. $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$,
4. $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$,
5. $(a, \infty) = \{x \in \mathbb{R} : x > a\}$,
6. $[a, \infty) = \{x \in \mathbb{R} : x \geq a\}$,
7. $(-\infty, b) = \{x \in \mathbb{R} : x < b\}$,
8. $(-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$.

Note that $(a, a) = [a, a) = (a, a] = \emptyset$ and $[a, a] = \{a\}$. Sets of the form (a, b), (a, ∞) and $(-\infty, a)$ are called open intervals. Whereas sets of the form $[a, b]$ are called closed intervals.

3.1 Definitions of supremum and infimum

We now introduce the notion of the supremum and infimum of sets. Note that the definitions crucially make use of the completeness axiom.

Definition 3.2 (Supremum)**.** *Let $A \subseteq \mathbb{R}$.*

1. *If A is not bounded above then we define the supremum of A to be infinity, that is $\sup A = \infty$.*

2. *If A is non-empty and bounded above we define the supremum of A, $\sup A$, to be the least upper bound for A. That is $\sup A = \alpha$ where $\alpha \in \mathbb{R}$ if and only if for all $a \in A$ we have that $a \leq \alpha$ and if $y \in \mathbb{R}$ with $y < \alpha$ there exists $a \in A$ with $a > y$.*

3. *If $A = \emptyset$ then we define the supremum of A to be minus infinity, i.e. $\sup A = -\infty$.*

Definition 3.3 (Infimum)**.** *Let $A \subseteq \mathbb{R}$.*

1. *If A is not bounded below then we define the infimum of A to be minus infinity, i.e. $\inf A = -\infty$.*

2. *If A is non-empty and bounded below we define the infimum of A, $\inf A$, to be the greatest lower bound for A. That is $\inf A = \alpha$ for $\alpha \in \mathbb{R}$ if and only if for all $a \in A$ we have that $a \geq \alpha$ and for any $y \in \mathbb{R}$ with $y > \alpha$ there exists $a \in A$ with $a < y$.*

3. *If $A = \emptyset$ then we define the infimum of A to be infinity,* $\inf A = \infty$.

Remark. *For $A \subseteq \mathbb{R}$ to show that $\sup A = \alpha$ where $\alpha \in \mathbb{R}$ there are two steps. Firstly you need to show that for all $x \in A$ we have that $x \leq \alpha$ (i.e. α is an upper bound. Secondly you need to show that for any $\beta < \alpha$ there exists $a \in A$ with $a > \beta$ (i.e. there is no upper bound smaller than α). An equivalent way to approach the second part is to show that for any $\epsilon > 0$ there exists $a \in A$ such that $a > \alpha - \epsilon$ (again this shows there is no upper bound smaller than α).*

We will illustrate this with the following very straightforward result.

Proposition 3.4. *Let $a, b \in \mathbb{R}$ with $a < b$. We have that:*

1. $\sup((a, b)) = \sup((a, b]) = \sup([b, a)) = \sup([a, b]) = b$,

2. $\inf((a, b)) = \inf([a, b)) = \inf((a, b]) = \inf([a, b]) = a$,

3. $\sup((a, \infty)) = \sup([a, \infty)) = \infty$,

4. $\inf((-\infty, a)) = \inf((-\infty, a)) = -\infty$,

5. $\sup((-\infty, a)) = \sup((-\infty, a]) = \inf((a, \infty)) = \inf([a, \infty)) = a$.

Proof. We will let $a, b \in \mathbb{R}$ with $a < b$ and show that $\sup((a, b)) = b$. **There are two steps needed here firstly we have to show for all $x \in (a, b)$ $x \leq b$ and secondly if $y < b$ then there exists $z \in (a, b)$ with $z > y$.** The first part is easy since if $x \in (a, b)$ then $a < x < b$ and so $x \leq b$. For the second step we need to fix $y \in \mathbb{R}$ with $y < b$. We first suppose that $y > a$. In this case $a < y < y + (b - y)/2 < b$ and so $y + (b - y)/2 > y$ and

$y+(b-y)/2 \in (a,b)$. Now suppose that $y \le a$ in which case $y < a+(b-a)/2$ and $a+(b-a)/2 \in A$.

We will now let $a \in \mathbb{R}$ and show that $\sup(a,\infty) = \infty$. To do this we need to show (a,∞) is not bounded above. **To show this we need to prove that for all $x \in \mathbb{R}$ there exists $y \in (a,\infty)$ such that $y > x$.** So we fix $x \in \mathbb{R}$ and note that $x \le |x|$ and so $x < |a|+|x|+1$. However $|a|+|x|+1 > a$ so $|a|+|x|+1 > x$ and $|a|+|x|+1 \in (a,\infty)$.

The other parts of the proposition can all be proved in a similar manner to one of the above two cases. □

We now give some examples of computing suprema and infima. Two of which are fairly concrete and one which is more theoretical.

3.2 Calculating suprema and infima

Define

$$A = \left\{ \frac{n+2}{n} : n \in \mathbb{N} \right\} \subseteq \mathbb{R}.$$

We will show that $\sup A = 3$ and $\inf A = 1$. For the supremum
1st step: We need to show for all $a \in A$, $a \le 3$. If $a \in A$ then $a = \frac{n+2}{n}$ for some $n \in \mathbb{N}$ and

$$\frac{n+2}{n} = 1 + \frac{2}{n} \le 1 + \frac{2}{1} = 3.$$

2nd step: We need to show that if $y < 3$ then y is not an upper bound for A. Note that taking $n = 1$ gives that $3 \in A$ so if $y < 3$ then y cannot be an upper for A.

Now for the infimum
1st step We need to show that for all $a \in A$, $a \ge 1$. If $a \in A$ then $a = \frac{n+2}{n}$ for some $n \in \mathbb{N}$ and

$$\frac{n+2}{n} = 1 + \frac{2}{n} \ge 1 + 0 = 1.$$

2nd step We need to show for any $y > 1$ we can find $a \in A$ with $a < y$. Let $y > 1$. Note that $\frac{2}{y-1}$ is a non-negative real number. So by the Archimedean principle we can find $n \in \mathbb{N}$ such that $n > \frac{2}{y-1}$ this means that

$$\frac{n+2}{n} = 1 + \frac{2}{n} < 1 + 2\frac{y-1}{2} = y.$$

Thus we have shown that there is an $a \in A$ with $a < y$ which is what we wanted. **You should ask yourself why did we choose n in this way.**

Define

$$B = \left\{ \frac{n^2+1}{|n+1/2|} : n \in \mathbb{Z} \right\}.$$

We will show that $\sup B = \infty$ and $\inf B = 4/3$. For the supremum:
1st and only step: We need to show that B is not bounded above. That is that for all $x \in \mathbb{R}$ we can find $b \in B$ with $b \geq x$. Firstly note that for $n \in \mathbb{N}$

$$\frac{n^2+1}{n+1/2} = \frac{1+n^{-2}}{n^{-1}+n^{-2}/2} \geq \frac{1}{2n^{-1}} = \frac{n}{2}.$$

Thus if by the Archimedean principle we choose $n \in \mathbb{N}$ with $n > 2x$. We have that

$$x < \frac{n}{2} \leq \frac{n^2+1}{n+1/2}$$

and since $\frac{n^2+1}{n+1/2} \in B$ we have shown that B is not bounded.

Now for the infimum.
1st step We need to show that for all $b \in B$, $b \geq 4/3$. So we let $b \in B$ and note that this means $b = \frac{n^2+1}{|n+\frac{1}{2}|}$ for some $n \in \mathbb{Z}$. We have that by the triangle inequality

$$n^2 + 1 - 4/3(|n+\frac{1}{2}|) \geq n^2 + 1 - 4/3(|n| + \frac{1}{2}) = n^2 - 4/3|n| + 1/3.$$

By completing the square we get

$$n^2 - 4/3|n| - 2/3 = (|n| - 2/3)^2 - 1/9.$$

Since $(|n| - 2/3)^2 \geq 1/9$ for all $n \in \mathbb{Z}$ we get that $n^2 - 4/3|n| - 2/3 \geq 0$ for all $n \in \mathbb{N}$. Thus for all $n \in \mathbb{Z}$ $n^2 + 1 - 4/3(|n+\frac{1}{2}|) \geq 0$ and so $\frac{n^2+1}{|n+\frac{1}{2}|} \geq 4/3$ which is what we wanted.
2nd step: For $n = 1$ we have that $\frac{n^2+1}{|n+\frac{1}{2}|} = \frac{4}{3}$. So $\frac{4}{3} \in B$ and no value $y > 4/3$ can be a lower bound for B. Therefore $\inf B = \frac{4}{3}$.

For our third and more theoretical examples we will let A_1, A_2 be bounded non-empty subsets of $\mathbb{R}$ and define

$$A_1 + A_2 = \{x + y : x \in A_1 \text{ and } y \in A_2\}.$$

In this case we will show that $\sup(A_1 + A_2) = \sup A_1 + \sup A_2$.

1st step: If we let $\alpha = \sup A_1$ and $\beta = \sup A_2$ and let $z \in A_1 + A_2$ then we know we can find $x \in A_1$ and $y \in A_2$ such that $x + y = z$. Thus $x \leq \alpha$ and $y \leq \beta$ and so $z = x + y \leq \alpha + \beta$. Thus we know that $A_1 + A_2$ is bounded above by $\alpha + \beta$.
2nd step: Now let $\epsilon > 0$ (we want to show that we can find $z \in A_1 + A_2$ where $z \in (\alpha+\beta-\epsilon, \alpha+\beta])$. We know that there must exist $a_1 \in (\alpha - \epsilon/2, \alpha)$ where $a_1 \in A_1$ (otherwise $\alpha - \epsilon/2$ would be an upper bound for A). There also must exists $a_2 \in (\beta - \epsilon/2, \beta)$ where $a_2 \in A_2$. So $a_1 + a_2 \in A_1 + A_2$ and $a_1 + a_2 \geq \alpha + \beta + \epsilon$. So we have shown that for any $\epsilon > 0$ $\alpha + \beta - \epsilon$ cannot be an upper bound for $A + B$. Thus since $\alpha + \beta$ is an upper bound for $A + B$ it must be the supremum of $A + B$.

Definition 3.5. *For* $A \subset \mathbb{R}$ *we say that for* $\alpha \in \mathbb{R}$, $\max A = \alpha$ *if and only if* $\sup A = \alpha$ *and* $\alpha \in A$. *Similarly for a subset* $A \subset \mathbb{R}$ *we say that for* $\alpha \in \mathbb{R}$, $\min A = \alpha$. *If and only if* $\inf A = \alpha$ *and* $\alpha \in A$. *Note that only certain subsets of the reals have maxima and similarly for minima.*

Example. *We take*

$$A = \left\{ \frac{n+2}{n} : n \in \mathbb{N} \right\} \subseteq \mathbb{R}.$$

We have already shown that $\sup A = 3$ *and* $\inf A = 1$. *We will now show that* A *has a maximum but has no minimum. To show that* A *has a maximum just consider* $n = 1$ *in which case* $\frac{1+2}{1} = 3 \in A$ *and so* $\sup A \in A$ *and we can see that* A *has a maximum.*

On the other hand for all $n \in \mathbb{N}$ $\frac{n+2}{n} = 1 + \frac{2}{n} > 1$. *So* A *does not contain* $1 = \inf A$ *and so* A *does not have a minimum.*

4 Sequences

4.1 Definition and basic properties

By a real valued sequence we will mean

$$(a_n)_{n\in\mathbb{N}} = (a_1, a_2, a_3, \ldots)$$

where for all $i \in \mathbb{N}$, $a_i \in \mathbb{R}$. The formal way of defining this would be as a function from the natural numbers to the real numbers. Note that a_i represents the ith element of the sequence and $(a_n)_{n\in\mathbb{N}}$ often shortened to (a_n) represents the whole sequence. The usual way to define a sequence will be by specifying a formula which a_n satisfies for all $n \in \mathbb{N}$.

Example. *Let $(a_n)_{n\in\mathbb{N}}$ be the sequence where $a_n = (-1)^n$ for all $n \in \mathbb{N}$. In this case*

$$(a_n)_{n\in\mathbb{N}} = (-1, 1, -1, 1, -1, \ldots).$$

We would have that $a_7 = (-1)^7 = -1$.

Definition 4.1. *A sequence of real numbers $(a_n)_{n\in\mathbb{N}}$ is bounded if there exists $K > 0$ such that for all $n \in \mathbb{N}$, $|a_n| \leq K$. We say that $(a_n)_{n\in\mathbb{N}}$ is bounded below if there exists $K \in \mathbb{R}$ such that $a_n \geq K$ for all $n \in \mathbb{N}$ and bounded above if there exists $K \in \mathbb{R}$ such that $a_n \leq K$ for all $n \in \mathbb{N}$.*

Example. *The sequence $(a_n)_{n\in\mathbb{N}}$ given by $a_n = (-1)^n$ is bounded since for all $n \in \mathbb{N}$ we have that $|a_n| = 1$. On the other hand the sequence $(b_n)_{n\in\mathbb{N}}$ where $b_n = n$ is not bounded. To see this we need to show that for any $K > 0$ we can find $n \in \mathbb{N}$ such that $|b_n| > K$. So we let $K > 0$ and note that by the Archimedean principle we can find $n \in \mathbb{N}$ where $n > K$ and so $|b_n| = n > K$. Note that the sequence $(b_n)_{n\in\mathbb{N}}$ is bounded below by 1.*

4.2 Convergent sequences

We now give the most important definition in this unit.

Definition 4.2. *A sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers is convergent to $\alpha \in \mathbb{R}$ if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \geq N$ then we have that*

$$|a_n - \alpha| \leq \epsilon.$$

When a sequence of real numbers $(a_n)_{n\in\mathbb{N}}$ is convergent to $\alpha \in \mathbb{R}$ we say that the sequence is convergent and write $\lim_{n\to\infty} a_n = \alpha$.

This means that in order to show that a sequence $(a_n)_{n\in\mathbb{N}}$ converges to a real number α you need to show that whichever $\epsilon > 0$ you are given you can find $N \in \mathbb{N}$ such that whenever you have a natural number $n \geq N$ you have that $|a_n - \alpha| \leq \epsilon$.

Example. *We start with a trivial example. If we let* $c \in \mathbb{R}$ *and* $a_n = c$ *for all* $n \in \mathbb{N}$ *then* $(a_n)_{n\in\mathbb{N}}$ *is convergent with* $\lim_{n\to\infty} a_n = c$*. To see this let* $\epsilon > 0$ *and choose* $N = 1$*. We then have that if* $n \in \mathbb{N}$ *and* $n \geq 1$*,*

$$|a_n - c| = 0 \leq \epsilon.$$

Example. *Let* $a_n = \frac{1}{n}$ *for all* $n \in \mathbb{N}$*. We will show that* $\lim_{n\to\infty} a_n = 0$*. Let* $\epsilon > 0$ *and by the Archimedean principle choose* $N \in \mathbb{N}$ *with* $N > \frac{1}{\epsilon}$*. We have that for any* $n \in \mathbb{N}$ *with* $n \geq N$*,*

$$|a_n - 0| = \frac{1}{n} \leq \frac{1}{N} \leq \epsilon.$$

Thus we have shown that for any $\epsilon > 0$ *there exists* $N \in \mathbb{N}$ *such that for all natural numbers* $n \geq N$ *we have* $|a_n - 0| \leq \epsilon$*. Therefore* $(a_n)_{n\in\mathbb{N}}$ *is convergent with* $\lim_{n\to\infty} a_n = 0$*.*

Example. *Let* $a_n = \frac{n^2-1}{n^2+1}$ *for all* $n \in \mathbb{N}$*. We will show that* $\lim_{n\to\infty} a_n = 1$*. Let* $\epsilon > 0$ *and by the Archimedean principle choose* $N \in \mathbb{N}$ *such that* $N > \frac{2}{\sqrt{\epsilon}}$*. We have that for any* $n \geq N$*,*

$$|a_n - 1| = \left|\frac{n^2-1}{n^2+1} - 1\right| = \left|\frac{n^2-1-(n^2+1)}{n^2+1}\right| = \frac{2}{n^2+1} \leq \frac{2}{n^2+1} \leq \frac{2}{N^2} \leq \epsilon.$$

Therefore $(a_n)_{n\in\mathbb{N}}$ *is convergent with* $\lim_{n\to\infty} a_n = 1$*.*

Remark. *In the second example before writing it down as written above it is necessary to do some work to figure out how to choose* N*. This is standard with these arguments while they should be written in the form 'Let* $\epsilon > 0$ *and choose* $N = \cdots$*' it is usually necessary to do some rough work to figure out how to choose* N *before writing this down.*

We can use the definition of a limit to say exactly what we mean by a sequence being convergent or divergent.

Definition 4.3. *Let* $(a_n)_{n\in\mathbb{N}}$ *be a real valued sequence we say that* $(a_n)_{n\in\mathbb{N}}$ *is convergent if and only if there exists* $\alpha \in \mathbb{R}$ *where* $\lim_{n\to\infty} a_n = \alpha$*. We say that a sequence* $(a_n)_{n\in\mathbb{N}}$ *is divergent if it is not convergent.*

To show that a sequence is divergent means that we have to show it is not convergent to any $\alpha \in \mathbb{R}$. So by negating the definition of a limit we have that a sequence $(a_n)_{n\in\mathbb{N}}$ is divergent if for any $\alpha \in \mathbb{R}$ there exists $\epsilon > 0$ such that for all $N \in \mathbb{N}$ there exists $n \in \mathbb{N}$ with $n \geq N$ where $|a_n - \alpha| > \epsilon$. This is quite a clunky process and in practice we won't often use this definition directly (instead we show that the sequence does not satisfy some property that all convergent sequences satisfy). However we will now give a couple of examples working directly from the definition.

Examples. *Let $(a_n)_{n\in\mathbb{N}}$ be the sequence given by $a_n = (-1)^n$ for all $n \in \mathbb{N}$. We will show that $(a_n)_{n\in\mathbb{N}}$ is divergent. Let $\alpha \in \mathbb{R}$ and choose $\epsilon = 1/2$. We have that for any $N \in \mathbb{N}$ that $|a_{N+1} - a_N| = 2$. So $2 = |a_{N+1} - a_N| \leq |a_{N+1} - \alpha| + |a_N - \alpha|$. Thus we must have that either $|a_{N+1} - \alpha| > \frac{1}{2} = \epsilon$ or $|a_N - \alpha| > \frac{1}{2} = \epsilon$. Either way we have shown that for all $\alpha \in \mathbb{R}$ for $\epsilon = \frac{1}{2}$ and for any $\mathbb{N} \in \mathbb{N}$ that we can find $n \in \mathbb{N}$ with $n \geq N$ and $|a_n - \alpha| > \epsilon$ (either take $n = N$ or $n = N + 1$.)*

Let $(b_n)_{n\in\mathbb{N}}$ be the sequence given by $b_n = \sqrt{n}$ for all $n \in \mathbb{N}$. We will show that $(b_n)_{n\in\mathbb{N}}$ is divergent. Let $\alpha \in \mathbb{R}$ and choose $\epsilon = 1$. By the Archimedean property we can find $k \in \mathbb{N}$ with $k > (\alpha + 1)^2$. Thus if we let $N \in \mathbb{N}$ and take $n = \max\{N, k\}$ then we have that $n \geq N$ and $b_n = \sqrt{n} \geq \sqrt{k} > \alpha + 1$. Thus $|b_n - \alpha| > 1$ and so $(b_n)_{n\in\mathbb{N}}$ is divergent.

4.3 Basic properties of convergent sequences

In this section we look at some simple consequences of a sequence being convergent. First of all a convergent sequence is bounded.

Theorem 4.4. *If $(a_n)_{n\in\mathbb{N}}$ is a convergent sequence of real numbers then (a_n) is bounded*

Proof. Since (a_n) is convergent we know there exists $\alpha \in \mathbb{R}$ with $\lim_{n\to\infty} a_n = \alpha$. Taking $\epsilon = 1$ in the definition of convergence this means there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - \alpha| \leq 1$. Now let $K = \max\{|a_i| : 1 \leq i \leq N\}$. For $n \leq N$ we have that by definition $|a_n| \leq K$. If $n \geq N$ we have by the triangle inequality

$$|a_n| = |a_n - \alpha + \alpha - a_N + a_N| \leq |a_n - \alpha| + |\alpha - a_N| + |a_N| \leq K + 2.$$

Therefore for all $n \in \mathbb{N}$ we have that $|a_n| \leq K + 2$. Thus $(a_n)_{n\in\mathbb{N}}$ is bounded. □

This means that any unbounded sequence must be divergent. However it is possible to have a bounded sequence which is not convergent. For example we have already seen that $a_n = (-1)^n$ for all $n \in \mathbb{N}$ is divergent but $|a_n| = 1$ for all $n \in \mathbb{N}$ so we can see that the sequence is bounded.

Secondly the limit of a sequence may only take one value

Proposition 4.5. *If $(a_n)_{n\in\mathbb{N}}$ is a convergent sequence and $a, b \in \mathbb{R}$ with $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} a_n = b$ then $a = b$.*

Proof. Suppose that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} a_n = b$. We will show that for all $\epsilon > 0$, $|a - b| \leq \epsilon$ and thus $a = b$. Let $\epsilon > 0$. We can choose $N_1 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N_1$ we have that $|a_n - a| \leq \epsilon/2$. We can also choose $N_2 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N_2$ we have that

$|a_n - b| \leq \epsilon/2$. Now take $N = \max\{N_1, N_2\}$ and note that by the triangle inequality,

$$|a - b| = |a - a_N + a_N - b| \leq |a - a_N| + |a_N - b| \leq \epsilon/2 + \epsilon/2 = \epsilon.$$

Thus for all $\epsilon > 0$ we know that $|a - b| \leq \epsilon$ and so $a = b$. □

We now give a couple of equivalent definitions of convergence which can be useful.

Note: The following statement and subsequent proof have been slightly modified from the original notes.

Proposition 4.6. *Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of real numbers and $\alpha \in \mathbb{R}$. The following statements are equivalent.*

(1) For all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$, we have

$$|a_n - \alpha| \leq \epsilon.$$

(2) Take $K > 0$; for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$, we have

$$|a_n - \alpha| \leq K\epsilon.$$

(3) For all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$, we have

$$|a_n - \alpha| < \epsilon.$$

Proof. To show all the statements are equivalent we will show that (1) implies (2) that (2) implies (3) and then (3) implies (1).

To see that (1) implies (2), we first reformulate (1) by renaming ε as ε'. So now (1) states: For all $\epsilon' > 0$, there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$, we have

$$|a_n - \alpha| \leq \epsilon'.$$

Suppose that (1) holds; take $K > 0$ and $\epsilon > 0$. Set $\varepsilon' = K\varepsilon$ (so $\epsilon' > 0$). Since $\lim_{n\to\infty} a_n = \alpha$, there is some $N \in \mathbb{N}$ so that for all $n \in \mathbb{N}$ with $n \geq N$ we have $|a_n - \alpha| \leq \varepsilon' = K\varepsilon$.

To see that (2) implies (3), we suppose that (2) holds. Take $\epsilon > 0$. Then with $K = 1/2$, (2) gives us that there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$, we have

$$|a_n - \alpha| \leq K\epsilon = \frac{\epsilon}{2} < \epsilon.$$

That (3) implies (2) is immediate, for whenever we have $|a_n - \alpha| < \epsilon$ we also have $|a_n - \alpha| \leq \epsilon$. □

We now state a theorem which is extremely useful for finding the limits of certain sequences without working directly from the definition.

Theorem 4.7 (Sandwich rule)**.** *Let $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}, (c_n)_{n\in\mathbb{N}}$ be real valued sequences and $\alpha \in \mathbb{R}$. If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = \alpha$ and for all $n \in \mathbb{N}$ we have that $a_n \leq b_n \leq c_n$ then we have that $(b_n)_{n\in\mathbb{N}}$ is convergent and $\lim_{n\to\infty} b_n = \alpha$.*

Proof. Let $\epsilon > 0$ and choose $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - \alpha| \leq \epsilon/3$ and $|c_n - \alpha| \leq \epsilon/3$. We then have that for all $n \geq N$ by the triangle inequality and the fact that $a_n \leq b_n \leq c_n$ (which means that $|b_n - a_n| \leq |c_n - a_n|$),

$$\begin{aligned} |b_n - \alpha| &= |b_n - a_n + a_n - \alpha| \leq |b_n - a_n| + |a_n - \alpha| \\ &\leq |c_n - a_n| + |a_n - \alpha| \leq |c_n - \alpha| + |\alpha - a_n| + |a_n - \alpha| \\ &= |c_n - \alpha| + 2|a_n - \alpha| \leq \epsilon. \end{aligned}$$

□

Remark. *Similarly we also have that for two sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ that if $a_n \leq b_n$ for all $n \in \mathbb{N}$, $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$ then $a \leq b$, as shown in the 4th exercise sheet.*

We complete the section with a couple of examples using the sandwich rule.

Example. *If we take $(b_n)_{n\in\mathbb{N}}$ to be the sequence with $b_n = \frac{1}{5n+6}$ for all $n \in \mathbb{N}$ we have that for all $n \in \mathbb{N}$ $0 \leq \frac{1}{5n+6} \leq \frac{1}{n}$. We take (a_n) to be the sequence with $a_n = 0$ for all $n \in \mathbb{N}$ and (c_n) to be the sequence with $c_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. We then have that $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = 0$. Thus by the sandwich rule $\lim_{n\to\infty} b_n = 0$.*

Corollary 4.8. *For any $k \in \mathbb{N}$, the sequence $(a_n)_{n\in\mathbb{N}}$ defined by $a_n = n^{-k}$ for all $n \in \mathbb{N}$ is convergent with $\lim_{n\to\infty} a_n = 0$.*

Proof. Let $k \in \mathbb{N}$ we then have that $0 \leq n^{-k} \leq n^{-1}$ and since $\lim_{n\to\infty} 0 = 0$ and $\lim_{n\to\infty} n^{-1} = 0$ the result follows by the sandwich rule. □

4.4 Arithmetic properties of limits

Always working directly from the definition of a limit can be a long process. However fortunately there are several results which we can use to make things a much quicker process. We have already seen one example with the sandwich rule. Here is another result which will help us do this.

Theorem 4.9. *Let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ be real valued sequences and let $a, b, c \in \mathbb{R}$. Suppose that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. We have:*

1. ***Sum rule:*** $\lim_{n\to\infty} a_n + b_n = a + b$,
2. ***Product rule:*** $\lim_{n\to\infty} a_n b_n = ab$,

3. ***Scalar product rule:*** $\lim_{n\to\infty} ca_n = ca$,

4. ***Quotient rule:*** *If* $b \neq 0$ *and* $b_n \neq 0$ *for all* $n \in \mathbb{N}$ *then* $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$.

Proof. **Sum rule**: Let $\epsilon > 0$. Since we have that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$ we can choose $N \in \mathbb{N}$ such that if $n \geq N$ then $|a_n - a| \leq \epsilon/2$ and $|b_n - b| \leq \epsilon/2$. Now if $n \geq n$ by the triangle inequality we have

$$|a_n + b_n - (a+b)| \leq |a_n - a| + |b_n - b| \leq \epsilon/2 + \epsilon/2 = \epsilon.$$

Thus $\lim_{n\to\infty} a_n + b_n = a + b$.
Product rule: Since (a_n) and (b_n) are both convergent we know that they are both bounded. So there exists $K > 0$ such that for all $n \in \mathbb{N}$, $|b_n| \leq K$ and $|a| \leq K$. Now let $\epsilon > 0$ and choose $N \in \mathbb{N}$ such that if $n \geq N$ $|a_n - a| \leq \frac{\epsilon}{2K}$ and $|b_n - b| \leq \frac{\epsilon}{2K}$. We then have that if $n \geq N$ by the triangle inequality

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - ab_n + ab_n - ab| \leq |a_n b_n - ab_n| + |ab_n - ab| \\ &= |b_n||a_n - a| + |a||b_n - b| \leq K(|a_n - a| + |b_n - b|) \leq \epsilon. \end{aligned}$$

Scalar product rule If we take $(c_n)_{n\in\mathbb{N}}$ to be the sequence with $c_n = c$ for all $n \in \mathbb{N}$ then $\lim_n \to \infty c_n = c$ and so by the product rule

$$\lim_{n\to\infty} ca_n = \lim_{n\to\infty} c_n a_n = ca.$$

Quotient rule We will let c_n be the sequence with $c_n = \frac{1}{b_n}$ for all $n \in \mathbb{N}$. We will first show that $\lim_{n\to\infty} c_n = \frac{1}{b}$. Since $b \neq 0$ and $\lim_{n\to\infty} b_n = b$ we can find $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$ we have that $|b_n - b| \leq |b|/2$ and so $|b_n| \geq |b|/2$. So we will let $\epsilon > 0$ and choose N such that for all $n \geq N$ $|b_n| \geq |b|/2$ and $|b_n - b| \leq \frac{\epsilon b^2}{2}$. Thus by the triangle inequality we have for $n \geq N$,

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \left|\frac{b_n - b}{b_n b}\right| = \frac{|b_n - b|}{|b_n b|} \leq \frac{|b_n - b|}{b^2/2} \leq \epsilon.$$

Thus $\lim_{n\to\infty} c_n = \frac{1}{b}$. Since we can write $\frac{a_n}{b_n} = a_n c_n$ it follows by the product rule that $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$.

□

Example. *Let* $(a_n)_{n\in\mathbb{N}}$ *satisfy that* $a_n = \frac{5n^3+6n+4}{2n^3+n^2+9}$ *for all* $n \in \mathbb{N}$. *We will show that* $\lim_{n\to\infty} a_n = \frac{5}{2}$. *We have that*

$$a_n = \frac{5n^3 + 6n + 4}{2n^3 + n^2 + 9} = \frac{5 + 6n^{-2} + 4n^{-3}}{2 + n^{-1} + 9n^{-3}}.$$

We have that by Corollary 4.8 $\lim_{n\to\infty} n^{-k} = 0$ *for all* $k \in \mathbb{N}$ *and we know that* $\lim_{n\to\infty} c = c$ *for all* $c \in \mathbb{R}$. *So by the scalar product rule and the sum rule* $\lim_{n\to\infty} 5 + 6n^{-2} + 4n^{-3} = 5$ *and* $\lim_{n\to\infty} 2 + n^{-1} + 9n^{-3} = 2$. *Moreover* $2 + n^{-1} + 9n^{-3} > 0$ *for all* $n \in \mathbb{N}$ *so by the quotient rule* $\lim_{n\to\infty} a_n = 5/2$.

Remark. *The example above illustrates a standard method when working with this type of sequence. That is when dealing with a sequence defined as a quotient, look for a dominant term (by this we mean the term in the numerator or dominator which we expect to grow at the fastest rate). You should then divide both numerator and denominator by this dominant term.*

4.5 Divergent sequences

We will now look at some special cases of divergent sequences. These are sequences which are divergent to $\pm\infty$.

Definition 4.10. *Let $(a_n)_{n\in\mathbb{N}}$ be a real valued sequence. We say that (a_n) is divergent to ∞ and write $\lim_{n\to\infty} a_n = \infty$ if and only if for all $K \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $a_n > K$. We say that (a_n) is divergent to $-\infty$ and write $\lim_{n\to\infty} a_n = -\infty$ if and only if for all $K \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $a_n < K$.*

Examples. *Consider the sequence $(a_n)_{n\in\mathbb{N}}$ where $a_n = \frac{n^2+1}{n+4}$ for all $n \in \mathbb{N}$. We will show that $\lim_{n\to\infty} a_n = \infty$. First of all*

$$\frac{n^2+1}{n+4} = \frac{1+n^{-2}}{n^{-1}+4n^{-2}} \geq \frac{1}{5n^{-1}} = \frac{n}{5}.$$

We now fix $K \in \mathbb{R}$ and note that by the Archimedean principle we can find $N \in \mathbb{N}$ with $N > 5K$. Then for all $n \geq N$,

$$\frac{n^2+1}{n+4} \geq \frac{n}{5} \geq \frac{N}{5} > K.$$

Thus $\lim_{n\to\infty} \frac{n^2+1}{n+4} = \infty$.

Now consider the sequence $(b_n)_{n\in\mathbb{N}}$ with $b_n = (-1)^n n^2$ for all $N \in \mathbb{N}$ we will have that b_n is nether divergent to ∞ or $-\infty$, however $|b_n|$ is divergent to infinity ($\lim_{n\to\infty} |b_n| = \infty$). To justify this note that if n is odd then $b_n \leq 0$ but if n is even then $b_n \geq 0$. Thus to see that (b_n) is not divergent to infinity take $K = 0$ then for any $N \in \mathbb{N}$ either $b_N \leq 0$ or $b_{N+1} \leq 0$ so it is not possible to find N such that for all $n \geq N$, $b_n > 0$ so (b_n) is not divergent to infinity. Similarly we can show (b_n) is not divergent to $-\infty$.

However $|b_n| = n^2$ and we can easily show this is divergent to infinity.

Proposition 4.11. *Let $(a_n)_{n\in\mathbb{N}}$ be a real valued sequence with $a_n \neq 0$ for any $n \in \mathbb{N}$ but $\lim_{n\to\infty} a_n = 0$. Let $b_n = a_n^{-1}$ for all $n \in \mathbb{N}$. We have that:*

1. *$\lim_{n\to\infty} |b_n| = \infty$.*
2. *If $a_n > 0$ for all $n \in \mathbb{N}$ then $\lim_{n\to\infty} b_n = \infty$.*
3. *If $a_n < 0$ for all $n \in \mathbb{N}$ then $\lim_{n\to\infty} b_n = -\infty$.*

Proof. 1. Let $x \in \mathbb{R}$ and choose $\epsilon = \frac{1}{|x|+1}$ (this is so that $\epsilon^{-1} > |x|$). Since $\lim_{n\to\infty} a_n = 0$ we can choose $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n| \leq \epsilon$. Thus for $n \geq N$,

$$|b_n| = \frac{1}{|a_n|} \geq \frac{1}{\epsilon} = |x| + 1 > |x| \geq x.$$

Thus $\lim_{n\to\infty} |b_n| = \infty$.

2. We simply need to note that in this case $b_n = |b_n|$ and so the result follows by the previous part.

3. We leave this case as an exercise (you simply need to show that if $\lim_{n\to\infty} x_n = \infty$ then $\lim_{n\to\infty} -x_n = -\infty$.)

□

Remark. *Several other properties of such sequences can be proved in similar ways. There are quite a few examples of this on the 5th exercise sheet.*