ANALYSIS 1A

Notes by Dr Thomas Jordan – Part 2 of 2

5 Monotonic sequences, subsequences and Cauchy sequences

5.1 Monotonic sequences

So far when trying to show a sequence is convergent we have first needed to figure out what the limit of the sequence is. However in several situations it can be hard to workout what the limit is going to be. Also sometimes we may want to define a number to be a limit (e is an example which is often defined in this way). This means it is useful to have criteria for when a sequence is convergent when we do not need to specify what the limit is. In this subsection we will give the first such criteria.

Definition 5.1. *We say that $(x_n)_{n\in\mathbb{N}}$, a sequence of real numbers, is monotone increasing if for all $n \in \mathbb{N}$ we have that $a_{n+1} \geq a_n$. Similarly we say that a sequence $(a_n)_{n\in\mathbb{N}}$ is monotone decreasing if for all $n \in \mathbb{N}$ we have that $a_{n+1} \leq a_n$.*

Remark. *If in fact we have that $a_{n+1} > a_n$ for all $n \in \mathbb{N}$ then we say that the sequence is strictly monotone increasing (with the obvious analogue if $a_{n+1} < a_n$ for all $n \in \mathbb{N}$). If we refer to a sequence as monotone this means it will be either monotone increasing or monotone decreasing.*

Remark. *If a sequence (a_n) is monotone increasing then it is easy to see by induction that for any $n, k \in \mathbb{N}$, $a_{n+k} \geq a_n$.*

Theorem 5.2 (Monotone Convergence Theorem)**.** *Let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence of real numbers.*

1. *If $(a_n)_{n\in\mathbb{N}}$ is a monotone increasing sequence then (a_n) is convergent and $\lim_{n\to\infty} a_n = \sup\{a_n : n \in \mathbb{N}\}$.*
2. *If $(a_n)_{n\in\mathbb{N}}$ is a monotone decreasing sequence then (a_n) is convergent and $\lim_{n\to\infty} a_n = \inf\{a_n : n \in \mathbb{N}\}$.*

Proof. We will just prove part 1 here. Part 2. can be deduced from part 1. by considering the sequence with terms $-a_n$.

For part 1. We let $\alpha = \sup\{a_n : n \in \mathbb{N}\}$ and note that since (a_n) is bounded, $\alpha \in \mathbb{R}$, we will show that $\lim_{n\to\infty} a_n = \alpha$. Let $\epsilon > 0$ and choose N such that $a_N > \alpha - \epsilon$ which we can do by the definition of α. For any $n \geq N$ we will have that $\alpha \geq a_n \geq a_N$ (since $a_{n+1} \geq a_n$ for any $n \in \mathbb{N}$). Thus $|a_n - \alpha| \leq |a_N - \alpha| \leq \epsilon$. □

Once we have shown a sequence is convergent we can sometimes use the following simple result to deduce what the limit must be.

Proposition 5.3. *Let $(a_n)_{n\in\mathbb{N}}$ be a convergent real valued sequence. We have that for any $k \in \mathbb{N}$ that $\lim_{n\to\infty} a_{n+k} = \lim_{n\to\infty} a_n$.*

Proof. See Exercise sheet 4 question 5. □

Corollary 5.4. *Let $\lambda \in \mathbb{R}$. We have that*

1. *If $|\lambda| < 1$ then $\lim_{n\to\infty} \lambda^n = 0$,*
2. *If $\lambda > 1$ then $\lim_{n\to\infty} \lambda^n = \infty$*
3. *If $\lambda < -1$ then $\lim_{n\to\infty} |\lambda^n| = \infty$.*

Proof. Parts 2. and 3. can be deduced from part 1 by using Proposition 4.11.

To prove part 1 first consider the case when $0 < \lambda < 1$. In this case $\lambda^n \geq 0$ and $\lambda^{n+1} < \lambda^n$ for all $n \in \mathbb{N}$. Thus the sequence $(\lambda^n)_{n\in\mathbb{N}}$ is monotone decreasing and bounded below and so convergent with $\lim_{n\to\infty} \lambda^n = \inf\{\lambda^n : n \in \mathbb{N}\}$. Moreover we have that by Proposition 5.3 and the scalar multiple rule that

$$\lambda \lim_{n\to\infty} \lambda^n = \lim_{n\to\infty} \lambda^{n+1} = \lim_{n\to\infty} \lambda^n.$$

Thus if we let $\alpha = \lim_{n\to\infty} \lambda^n$ then $\alpha = \lambda\alpha$. So $\alpha(1-\lambda) = 0$ and since $\lambda \neq 0$ we must have that $\alpha = 0$.

For $-1 < \lambda < 0$ we can use that if $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$ (this is an easy exercise) and that $|\lambda^n| = |\lambda|^n$. Finally the case when $\lambda = 0$ is obvious. □

We now give a couple of other examples of using the Monotone convergence theorem.

Example. *Let $(x_n)_{n\in\mathbb{N}}$ be the sequence where $x_1 = 1$ and $x_{n+1} = \frac{2x_n+1}{x_n+1}$ for all $n \in \mathbb{N}$. We will first show by induction that $0 \leq x_n \leq \frac{\sqrt{5}+1}{2}$. We know that this is true for $n = 1$ and we will suppose that it is true for some $n = k$. Then clearly $x_{k+1} \geq 0$. Moreover*

$$\begin{aligned} 2x_k + 1 - \frac{\sqrt{5}+1}{2}(x_k+1) &= x_k\left(2 - \frac{\sqrt{5}+1}{2}\right) - \left(\frac{\sqrt{5}+1}{2} - 1\right) \\ &\leq \frac{\sqrt{5}+1}{2}\left(2 - \frac{\sqrt{5}+1}{2}\right) - \left(\frac{\sqrt{5}+1}{2} - 1\right) \\ &= \frac{\sqrt{5}}{2} - \frac{1}{2} - \frac{\sqrt{5}}{2} + \frac{1}{2} = 0. \end{aligned}$$

Thus rearranging this we get that

$$x_{k+1} = \frac{2x_k + 1}{x_k + 1} \leq \frac{\sqrt{5}+1}{2}.$$

So it follows by induction that $x_n \leq \frac{\sqrt{5}+1}{2}$ *for all* $n \in \mathbb{N}$.

Now for $n \in \mathbb{N}$

$$x_{n+1} - x_n = \frac{2x_n + 1}{x_n + 1} - x_n = \frac{-x_n^2 + x_n + 1}{x_n + 1}$$

and since $-x_n^2 + x_n + 1 = -(x_n - (1+\sqrt{5})/2)(x_n - (1-\sqrt{5})/2)$ *we know that if* $0 \leq x_n \leq \frac{\sqrt{5}+1}{2}$ *then* $\frac{-x_n^2+x_n+1}{x_n+1} \geq 0$. *Thus* $x_{n+1} - x_n \geq 0$ *for all* $n \in \mathbb{N}$ *and so* $(x_n)_{n\in\mathbb{N}}$ *is monotone increasing. Now let* $\alpha = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1}$. *By the arithmetic properties of limits* $\alpha = \frac{2\alpha+1}{\alpha+1}$ *and so* $\alpha^2 - \alpha - 1 = 0$ *and* $\alpha = (\sqrt{5}+1)/2$ *or* $\alpha = (1-\sqrt{5})/2$. *However since* $x_n \geq 0$ *for all* $n \in \mathbb{N}$ *we can conclude that* $\alpha = \frac{\sqrt{5}+1}{2}$.

The next example gives one way of defining e.

Example. *Let* $a_n = (1 + 1/n)^n$. *We will show that* $(a_n)_{n\in\mathbb{N}}$ *is convergent.*
Claim: *For all* $n \in \mathbb{N}$, $a_n \leq 3$.
We use the binomial theorem to get that

$$(1 + 1/n)^n = \sum_{k=0}^{n} \binom{n}{k} n^{-k}.$$

Moreover

$$\binom{n}{k} n^{-k} = \frac{n \cdots (n-k+1)}{k!} n^{-k} = \frac{nn^{-1} \cdots (n-k+1)n^{-1}}{k!} \leq \frac{1}{k!}.$$

So since for all $k \in \mathbb{N}$, $(k+1)! \geq 2^k$ *we have that*

$$(1 + 1/n)^n \leq \sum_{k=0}^{n} \frac{1}{k!} \leq 2 + \sum_{k=1}^{n} 2^{-k}.$$

Now $\sum_{k=1}^{n} 2^{-n} = 1 - 2^{-n}$ *(it's just a geometric progression) and since* $1 - 2^{-n} \leq 1$ *for all* $n \in \mathbb{N}$ *we get that,*

$$(1 + 1/n)^n \leq 2 + 1 - 2^{-n} \leq 3.$$

Claim: *For all* $n \in \mathbb{N}$, $a_{n+1} \geq a_n$
Again by using the Binomial Theorem we have that

$$\begin{aligned} a_{n+1} - a_n &= \sum_{k=0}^{n+1} \binom{n+1}{k} (n+1)^{-k} - \sum_{k=0}^{n} \binom{n}{k} n^{-k} \\ &= (n+1)^{-(n+1)} + \sum_{k=0}^{n} \left(\binom{n+1}{k} (n+1)^{-k} - \binom{n}{k} n^{-k} \right). \end{aligned}$$

Now we have that

$$\binom{n}{k} n^{-k} = \frac{n \cdots (n-k+1) n^{-k}}{k!} = \frac{1 \cdot (1-1/n) \cdots (1-(k-1)/n)}{k!}$$

and

$$\begin{aligned} \binom{n+1}{k} (n+1)^{-k} &= \frac{(n+1) \cdots ((n+1)-(k-1))(n+1)^{-k}}{k!} \\ &= \frac{1 \cdot (1-1/(n+1)) \cdots (1-(k-1)/(n+1))}{k!}. \end{aligned}$$

By comparing these expressions we can see that the denominators are both $k!$ and the numerators are both the product of k terms and in the second expression each of the terms is greater than or equal to the corresponding term in the first expression. Thus

$$\binom{n}{k} n^{-k} \leq \binom{n+1}{k} (n+1)^{-k}$$

and we can conclude that

$$a_{n+1} - a_n = (n+1)^{-(n+1)} + \sum_{k=0}^{n} \left(\binom{n+1}{k} (n+1)^{-k} - \binom{n}{k} n^{-k} \right) \geq 0.$$

Thus we have shown that (a_n) is bounded above and monotone increasing. So (a_n) is convergent. In fact one way of defining e is as $\lim_{n\to\infty}(n+n^{-1})^n$.

5.2 Subsequences and the Bolzano-Weierstrass Theorem

Definition 5.5. *Let $(a_n)_{n\in\mathbb{N}}$ be a real valued sequence. If $(n_k)_{n\in\mathbb{N}}$ is a strictly monotone increasing sequence of natural numbers (so $n_1 < n_2 < n_3 < \cdots$) then we say that*

$$(a_{n_k})_{k\in\mathbb{N}} = (a_{n_1}, a_{n_2}, a_{n_3}, \ldots)$$

is a subsequence of $(a_n)_{n\in\mathbb{N}}$.

Remark. *Suppose that $(n_k)_{k\in\mathbb{N}}$ is a strictly monotone increasing sequence of natural numbers. Then (as one can prove by induction) for every $k \in \mathbb{N}$ we have $n_k \geq k$. Also, if $(a_{n_k})_{k\in\mathbb{N}}$ is a subsequence of a bounded sequence $(a_n)_{n\in\mathbb{N}}$, then $(a_{n_k})_{k\in\mathbb{N}}$ is also a bounded sequence.*

Example. *If we consider the sequence $(a_n)_{n\in\mathbb{N}}$ then $(a_{2n})_{n\in\mathbb{N}}$ is the subsequence of (a_n) consisting of the even terms. So if $a_n = \frac{n(-1)^n+7}{n}$ for all $n \in \mathbb{N}$ then*

$$(a_{2n})_{n\in\mathbb{N}} = (9/2, 11/4, 13/6, 15/8, \ldots).$$

We now give a couple of results relating the properties of sequence to subsequences.

Proposition 5.6. *Let $(a_n)_{n\in\mathbb{N}}$ be a real valued sequence.*

1. *If $(a_n)_{n\in\mathbb{N}}$ is convergent with $\lim_{n\to\infty} a_n = \alpha$ where $\alpha \in \mathbb{R}$ then for all subsequences of (a_n), $(a_{n_k})_{k\in\mathbb{N}}$ we have that (a_{n_k}) is convergent with $\lim_{k\to\infty} a_{n_k} = \alpha$.*

2. *If $(a_n)_{n\in\mathbb{N}}$ satisfies $\lim_{n\to\infty} |a_n| = \infty$ then for all subsequences of (a_n), $(a_{n_k})_{k\in\mathbb{N}}$ we have that (a_{n_k}) satisfies $\lim_{k\to\infty} |a_{n_k}| = \infty$.*

3. *If $(a_n)_{n\in\mathbb{N}}$ satisfies $\lim_{n\to\infty} a_n = \infty$ then for all subsequences of (a_n), $(a_{n_k})_{k\in\mathbb{N}}$ we have that (a_{n_k}) satisfies $\lim_{k\to\infty} a_{n_k} = \infty$.*

4. *If $(a_n)_{n\in\mathbb{N}}$ satisfies $\lim_{n\to\infty} a_n = -\infty$ then for all subsequences of (a_n), $(a_{n_k})_{k\in\mathbb{N}}$ we have that (a_{n_k}) satisfies $\lim_{k\to\infty} a_{n_k} = -\infty$.*

Proof. We will just prove the first two parts of the proposition the final two parts can be proved very similarly. For the first part let (a_{n_k}) be a subsequence of (a_n) and $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that if $n \geq N$ and $n \in \mathbb{N}$ then $|a_n - \alpha| \leq \epsilon$. Now let $k \in \mathbb{N}$ with $k \geq N$. Since n_k is a strictly increasing sequence of natural numbers we must have that $n_k \geq k \geq N$ and so

$$|a_{n_k} - \alpha| \leq \epsilon.$$

Thus $\lim_{k\to\infty} a_{n_k} = \alpha$.

For the second point let (a_{n_k}) be a subsequence of (a_n) and $x \in \mathbb{R}$. Choose $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$ then $|a_n| > x$. Now take $k \geq N$ and then since $n_k \geq k \geq N$ we will have that $|a_{n_k}| \geq x$. So $\lim_{k\to\infty} |a_{n_k}| = \infty$. □

The converse of the statements in this proposition are not true. If $(a_n)_{n\in\mathbb{N}}$ is divergent it certainly does not imply that any subsequence of a_n will be divergent. For example let $a_n = (-1)^n$ for all $n \in \mathbb{N}$. We have already seen that (a_n) is divergent however $a_{2k} = 1$ for all $k \in \mathbb{N}$ so $(a_{2k})_{k\in\mathbb{N}}$ is convergent. In fact showing a sequence has two subsequences converging to different values is a good way of showing that a sequence is divergent.

Proposition 5.7. *Let $(a_n)_{n\in\mathbb{N}}$ be an unbounded sequence. There exists a subsequence (a_{n_k}) of (a_n) where $\lim_{k\to\infty} |a_{n_k}| = \infty$.*

Proof. We will construct our sequence inductively as follows. Let

$$n_1 = \min\{n : |a_n| \geq 1\}$$

and for $k \in \mathbb{N}$ let

$$n_{k+1} = \min\{n \in \mathbb{N} : \ |a_n| \geq k+1 \text{ and } n > n_k \ \}$$

(note that since (a_n) is unbounded n_{k+1} exists). Thus $(n_k)_{k\in\mathbb{N}}$ is a strictly monotone increasing sequence. Also, for any $x \in \mathbb{R}$, by the Archimedean Principle there is some $K \in \mathbb{N}$ so that $K \geq x - 1$; thus for all $k \in \mathbb{N}$ with $k \geq K$ we have $a_{n_k} \geq k \geq K > x$. Thus $\lim_{k\to\infty} |a_{n_k}| = \infty$. □

Note that we could have adjusted this construction so that $|a_{n_k}|$ would be monotone increasing by taking

$$n_{k+1} = \min\{n \in \mathbb{N} : \ |a_n| \geq k+1 \text{ and } |a_n| \geq |a_{n_k}| \text{ and } n > n_k \ \}.$$

We now do something similar for bounded sequences.

Note: The following definition and proof of theorem have been slightly modified from the original notes.

Definition 5.8. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. We say that an element a_n of $(a_n)_{n\in\mathbb{N}}$ is a peak if for all $m.n$ we have that $a_m < a_n$.*

Theorem 5.9 (Bolzano-Weierstrass Theorem)**.** *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. If the sequence (a_n) is bounded then (a_n) contains a convergent subsequence.*

Proof. To prove this we will show that any bounded sequence must contain a monotone subsequence and then apply the Monotone Convergence Theorem. This will be done by splitting the proof into two cases.

Case 1: We suppose that there exists a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$ such that each element a_{n_k} is a peak in $(a_n)n \in \mathbb{N}$. Thus for all $k \in \mathbb{N}$ and $m > n_k$ we know that $a_{n_k} > a_m$, and in particular, $a_{n_k} > a_{n_{k+1}}$. Therefore $(a_{n_k})_{k\in\mathbb{N}}$ is a monotone decreasing subsequence, and since it is a subsequence of a bounded sequence, it must be bounded. Therefore by the Monotone Convergence Theorem we know that $(a_{n_k})_{k\in\mathbb{N}}$ is convergent.

Case 2: We suppose that there is no subsequence (a_{n_k}) of (a_n) such that each element $(a_{n_k})_{k\in\mathbb{N}}$ is a peak in $(a_n)_{n\in\mathbb{N}}$. This means that the sequence $(a_n)_{n\in\mathbb{N}}$ can only have a finite number of peaks; therefore there exists $N \in \mathbb{N}$ such that for any $m \geq N$ we have that a_m is not a peak in $(a_n)_{n\in\mathbb{N}}$. (So for $m \geq N$, there is some $m' \in \mathbb{N}$ with $m' > m$ and $a_{m'} \geq a_m$.) We can therefore inductively construct a monotone increasing subsequence of $(a_n)_{n\in\mathbb{N}}$ as follows. Let $n_1 = N$. Therefore a_{n_1} is not a peak in $(a_n)_{n\in\mathbb{N}}$. Now suppose that $k \in \mathbb{N}$, and that $n_1, n_2, \ldots, n_k$ have been chosen so that $n_1 < n_2 < \cdots < n_k$ and $a_{n_1} \leq a_{n_2} \leq \cdots \leq a_{n_k}$. Set

$$n_k = \min\{n \in \mathbb{N} : \ n > n_k \text{ and } a_{n_k} \leq a_{n_{k+1}} \ \}.$$

In this way we produce a monotone increasing sequence $(a_{n_k})_{k\in\mathbb{N}}$ which is necessarily bounded since $(a_n)_{n\in\mathbb{N}}$ is bounded. Thus by the Monotone Convergence Theorem, $(a_{n_k})_{k\in\mathbb{N}}$ must converge (to some real number). □

5.3 Cauchy Sequences

Not every convergent sequence is monotone so while we know that a bounded monotone sequence is convergent it is not the case that all convergent sequence are monotone and bounded. However there is a definition for sequences which is equivalent to a sequence being convergent where we do not need to specify the limit. This is the notion of a Cauchy sequence.

Definition 5.10. *We say that a sequence of real numbers $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$ we have that $|x_n - x_m| \leq \epsilon$,*

We will start by giving some examples

Example. *Let $(a_n)_{n\in\mathbb{N}}$ satisfy that $a_n = \frac{2n}{n+1}$ for all $n \in \mathbb{N}$. We will show that $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Let $\epsilon > 0$ and choose $N \in \mathbb{N}$ by the Archimedean principle to be such that $N > \frac{4}{\epsilon}$. For $n, m \geq N$ we have that*

$$\left|\frac{2n}{n+1} - \frac{2m}{m+1}\right| = \left|\frac{2n-2m}{(m+1)(n+1)}\right|.$$

By the triangle inequality we get that

$$\left|\frac{2n-2m}{(m+1)(n+1)}\right| \leq \frac{2n}{(n+1)(m+1)} + \frac{2m}{(n+1)(m+1)} \leq \frac{2}{n+1} + \frac{2}{m+1} \leq \frac{4}{N} \leq \epsilon.$$

Let $(b_n)_{n\in\mathbb{N}}$ satisfy that $b_n = \frac{2^n}{2^n+1}$ for all $n \in \mathbb{N}$. We will show that $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Let $\epsilon > 0$, choose $N \in \mathbb{N}$ such that $2^{-(N-1)} \leq \epsilon$ (we can do this since we know that $\lim_{n\to\infty} 2^{-n} = 0$). For $n, m \geq N$ we have that

$$|a_n - a_m| = \left|\frac{2^n}{2^n+1} - \frac{2^m}{2^m+1}\right| = \left|\frac{2^n - 2^m}{(2^n+1)(2^m+1)}\right| \leq \left|\frac{2^n-2^m}{2^n 2^m}\right|.$$

By the triangle inequality we get

$$\left|\frac{2^n-2^m}{2^n 2^m}\right| \leq \frac{1}{2^m} + \frac{1}{2^n} \leq \frac{1}{2^{N-1}}.$$

Thus $|a_n - a_m| \leq \epsilon$.

We now give an example of a sequence which is not a Cauchy sequence.

Example. *Let $a_n = \frac{(-1)^n n+1}{n}$ for all $n \in \mathbb{N}$. We can see that for all $n \in \mathbb{N}$*

$$|a_{n+1} - a_n| = \left|\frac{(-1)^n n(n+1) - (-1)^{n+1} n(n+1) - 1}{n(n+1)}\right| \geq 2 - \frac{1}{n(n+1)} \geq 1$$

Thus if we take $\epsilon = \frac{1}{2}$ however we choose N we will always have that $|a_{N+1} - a_N| > \epsilon$ and so (a_n) cannot be a Cauchy sequence.

Proposition 5.11. *If $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence then it is a Cauchy sequence.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a convergence sequence and let $x = \lim_{n\to\infty} x_n$. We need to show that (x_n) is also a Cauchy sequence. Let $\epsilon > 0$ and use the fact that $x = \lim_{n\to\infty} x_n$ to choose N such that for all $n \geq N$ we have that $|x_n - x| \leq \epsilon/2$. If we let $n, m \geq N$ then using the triangle inequality

$$|x_n - x_m| = |x_n - x + x - x_m| \leq |x_n - x| + |x_m - x| \leq \epsilon.$$

Thus $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. □

It turns out that the converse is true as well.

Theorem 5.12 (Cauchy's principle of convergence)**.** *If $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence then it is convergent (i.e. there exists $x \in \mathbb{R}$ such that $\lim_{n\to\infty} x_n = x$.)*

Proof. This proof comes in three parts. The first part is to show that any Cauchy sequence is bounded, the second part is to show that this means any Cauchy sequence has a convergent subsequence (i.e. The Bolzano Weierstrass Theorem) and we can then show that any Cauchy sequence is convergent.

Step 1: We start by showing that any Cauchy sequence is bounded. To do this we use the fact that (x_n) is Cauchy to choose $N \in \mathbb{N}$ such that for all $n, m \geq N$ $|x_n - x_m| \leq 1$. We then let $K = \max\{|x_n| : 1 \leq n \leq N\}$ and so obviously $|x_i| \leq K$ for $1 \leq n \leq N$. On the other hand if $n > N$ then by the triangle inequality

$$|x_n| \leq |x_n - x_N + x_N| \leq |x_n - x_N| + |x_N| \leq 1 + K.$$

Thus for all $n \in \mathbb{N}$ we have $|x_n| \leq K + 1$.

Step 2: Since $(x_n)_{n\in\mathbb{N}}$ is bounded by the Bolzano-Weierstrass Theorem we know that it must contain a convergence subsequence x_{n_k}. We take such a subsequence x_{n_k} and let $x = \lim_{k\to\infty} x_{n_k}$.

Step 3: We can now complete the proof by showing $\lim_{n\to\infty} x_n = x$ to this we combine the fact that (x_n) is Cauchy with the fact that $x = \lim_{k\to\infty} x_{n_k}$. Let $\epsilon > 0$ and choose $N \in \mathbb{N}$ such that for all $n, m \geq N$ we have $|x_n - x_m| \leq \epsilon/2$. Since $x = \lim_{k\to\infty} x_{n_k}$ we can also find $k \in \mathbb{N}$ such that $n_k \geq N$ and $|x_{n_k} - x| \leq \epsilon/2$. Thus for $n \geq N$ we have that by the triangle inequality

$$|x_n - x| \leq |x_n - x_{n_k} + x_{n_k} - x| \leq |x_n - x_{n_k}| + |x_{n_k} - x| \leq \epsilon/2 + \epsilon/2 = \epsilon.$$

□

This gives us a method of showing a sequence is convergent when it is not monotone and we don't necessarily know what the limit is.

Example. *Let* $a_1 = 3$ *and* $a_{n+1} = 3 + \frac{1}{a_n}$*. We then have that for* $n \geq 2$

$$|a_{n+1} - a_n| = \left|3 + \frac{1}{a_n} - 3 + \frac{1}{a_{n-1}}\right| = \frac{|a_n - a_{n-1}|}{|a_n a_{n-1}|}.$$

Now we see that $a_n \geq 3$ *for all* $n \in \mathbb{N}$ *and so*

$$|a_{n+1} - a_n| \leq \frac{|a_n - a_{n-1}|}{9} \tag{1}$$

We can now show by induction that $|a_{n+1} - a_n| \leq \frac{1}{3 \cdot 9^{n-1}}$ *for all* $n \in \mathbb{N}$*. This is true for* $k = 1$ *since* $|a_2 - a_1| \leq \frac{1}{3}$ *and by inequality* (1) *we know that if it is true for* $n = k$ *then it is true for* $n = k + 1$*.*

We can now show that (a_n) *is a Cauchy sequence. Let* $\epsilon > 0$ *and choose* N *such that* $9^{-(N-1)} \leq \epsilon$*. For* $m > n \geq N$ *we have that by the triangle inequality*

$$|a_m - a_n| = \left|\sum_{k=n}^{m-1} (a_{k+1} - a_k)\right| \leq \sum_{k=n}^{m-1} |a_{k+1} - a_k| \leq \sum_{k=n}^{m-1} \frac{1}{3 \cdot 9^{k-1}}.$$

We have that by summing the geometric series

$$\sum_{k=n}^{m-1} \frac{1}{3 \cdot 9^{k-1}} = \frac{1}{3 \cdot 9^{n-1}} \sum_{k=0}^{m-n-1} 9^{-k} = \frac{1}{3 \cdot 9^{n-1}} \frac{9(1 - 9^{-(m-n)})}{8} \leq \frac{1}{9^{n-1}}.$$

Thus $|a_m - a_n| \leq 9^{-(N-1)} \leq \epsilon$ *and we have shown* $(a_n)_{n \in \mathbb{N}}$ *is Cauchy and so convergent.*

Now that we know the sequence is convergent we can find its limit. Since if $\lim_{n\to\infty} a_n = \alpha$ *we must have that since* $a_n \geq 3$ *for all* $n \in \mathbb{N}$*, that* $\alpha \geq 3$ *and by the arithmetic properties of limits* $\alpha = 3 + \frac{1}{\alpha}$*. So* $\alpha^2 - 3\alpha - 1 = 0$ *and we have that* $\alpha = \frac{3 \pm \sqrt{13}}{2}$*. Since* $\alpha \geq 3$ *we can deduce that* $\alpha = \frac{3 + \sqrt{13}}{2}$*.*

Remark. *Most of the material mentioned in this section was developed on the early to mid nineteenth century. You can read about three of the mathematicians involved at the website:*
http://www-history.mcs.st-andrews.ac.uk/
Here are brief biographies of Bolzano (1781-1848), Cauchy (1789-1857) and Weierstrass (1815-1897)
http://www-history.mcs.st-andrews.ac.uk/Mathematicians/Bolzano.html
http://www-history.mcs.st-andrews.ac.uk/Mathematicians/Cauchy.html
http://www-history.mcs.st-andrews.ac.uk/Mathematicians/Weierstrass.html

6 Series

If we have a real valued sequence $(a_n)_{n\in\mathbb{N}}$ then we can define a new sequence by letting

$$S_n = \sum_{k=1}^{n} a_k = a_1 + \cdots + a_n.$$

If the sequence S_n is convergent with limit S then we write $\sum_{n=1}^{\infty} a_n = S$. In general $\sum_{n=1}^{\infty} a_n$ is the infinite series associated to the sequence $(a_n)_{n\in\mathbb{N}}$. We make the following definition

Definition 6.1. *Let $(a_n)_{n\in\mathbb{N}}$ be a real valued sequence and $S_n = \sum_{k=1}^{n} a_k$.*

1. *If $(S_n)_{n\in\mathbb{N}}$ is a convergent sequence with $\lim_{n\to\infty} S_n = S$ then we say that the series $\sum_{n=1}^{\infty} a_n$ is a convergent series with $\sum_{n=1}^{\infty} a_n = S$.*
2. *If $(S_n)_{n\in\mathbb{N}}$ is a divergent sequence then we say that the series $\sum_{n=1}^{\infty} a_n$ is a divergent series and we cannot assign a real numbered value to it.*

So for a series $\sum_{n=1}^{\infty} a_n$ we have two sequences associated $(a_n)_{n\in\mathbb{N}}$ and the sequence of partial sums $(S_n)_{n\in\mathbb{N}}$. We will always use $(S_n)_{n\in\mathbb{N}}$ to denote the sequence of partial sums.

Examples. *We start with simple examples of a convergent series and a divergent series. Let $(a_k)_{k\in\mathbb{N}}$ be the sequence where $a_k = \frac{1}{k(k+1)}$, for all $k \in \mathbb{N}$. We then have that for $n \in \mathbb{N}$,*

$$S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) = 1 - \frac{1}{n+1}.$$

If you don't see why the last equality is true write out a few terms of $\sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right)$. Thus $S_n = 1 - \frac{1}{n+1}$ and so by the arithmetic properties of limits of sequences $\lim_{n\to\infty} S_n = 1$. Thus $\sum_{k=1}^{\infty} a_k$ is convergent and its value is 1.

Now let $b_n = 1$ for all $n \in \mathbb{N}$. In this case $S_n = n$ and is divergent. So $\sum_{n=1}^{\infty} b_n$ is a divergent series.

In the two examples above. We have been able to compute S_n explicitly, however this often will not be the case so it helps to have some simple results which give criteria for convergent or divergence. Our first such result gives a simple criteria for divergence.

Theorem 6.2. *Let $(a_n)_{n\in\mathbb{N}}$ be a real valued sequence. If the series $\sum_{n=1}^{\infty} a_n$ is convergent then $\lim_{n\to\infty} a_n = 0$*

Proof. Suppose that $\sum_{n=1}^{\infty} a_n$ is convergent. Then the sequence of partial sums S_n is convergent so we can let $S = \lim_{n\to\infty} S_n$ where $S \in \mathbb{R}$. We then have that for all $n \in \mathbb{N}$, $a_{n+1} = S_{n+1} - S_n$. Thus by the arithmetic properties of sequence and the fact that $\lim_{n\to\infty} S_{n+1} = S$ we get that $\lim_{n\to\infty} a_{n+1} = S - S = 0$ and so $\lim_{n\to\infty} a_n = 0$. □

So if we have a sequence $(a_n)_{n\in\mathbb{N}}$ which is either divergent or convergent but where the limit is not 0 we know that the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Example. *Let* $a_n = \frac{n}{n+5}$. *Since* $\lim_{n\to\infty} a_n = 1 \neq 0$ *we know that the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

However the converse of Theorem 6.2 is not true, there are examples of sequence (a_n) where $\lim_{n\to\infty} a_n = 0$ but the series $\sum_{n=1}^{\infty} a_n$ is divergent. Here is a particularly important example. is divergent.

Proposition 6.3. *The series* $\sum_{k=1}^{\infty} \frac{1}{k}$ *is divergent.*

Proof. We will consider the seuqence of partial sums $(S_n)_{n\in\mathbb{N}}$ and show that for each $n \in \mathbb{N}$ $S_{2^n-1} \geq n/2$ this will mean that the sequence $(S_n)_{n\in\mathbb{N}}$ is unbounded and so divergent. To do this we will use that for all $k \in \mathbb{N}$,

$$\sum_{i=2^k}^{2^{k+1}-1} \frac{1}{i} \geq \sum_{i=2^k}^{2^{k+1}-1} \frac{1}{2^{k+1}} = \frac{1}{2}.$$

Now we can write that for all $n \geq 2$,

$$S_{2^n-1} = 1 + \sum_{k=2}^{n} \sum_{i=2^{k-1}}^{2^k-1} \frac{1}{i} \geq 1 + (n-1)/2 \geq n/2.$$

Since $S_1 = 1 \geq \frac{1}{2}$ we have that for all $n \in \mathbb{N}$ $S_{2^n-1} \geq n/2$ and so the sequence of partial sums S_n is divergent and so the series $\sum_{k=1}^{\infty} \frac{1}{k}$ is divergent. In fact since $(S_n)_{n\in\mathbb{N}}$ is a monotone increasing sequence we have shown that $\lim_{n\to\infty} S_n = \infty$. □

So far we have a criteria to show that a series is divergent but to show that it is convergent our only method is to find $(S_n)_{n\in\mathbb{N}}$. We now want to give some methods of showing series are convergent. Our first step is the following result.

Proposition 6.4. *Let* $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ *be sequences,* $x, y \in \mathbb{R}$ *and* $m \geq 2$ *be a natural number.*

1. *If the series* $\sum_{n=1}^{\infty} a_n$ *and the series* $\sum_{n=1}^{\infty} b_n$ *are convergent then the series* $\sum_{n=1}^{\infty}(xa_n + yb_n)$ *is convergent.*

2. *The series* $\sum_{n=1}^{\infty} a_n$ *is convergent if and only if the series* $\sum_{n=m}^{\infty} a_n$ *is convergent.*

3. *If* $a_n \geq 0$ *for all* $n \in \mathbb{N}$ *and there exists* $M > 0$ *such that for all* $k \in \mathbb{N}$, $\sum_{n=1}^{k} a_n \leq M$ *then the series* $\sum_{n=1}^{\infty} a_n$ *is convergent.*

Proof. 1. We fix $k \in \mathbb{N}$ and note that

$$\sum_{n=1}^{k}(xa_n + by_n) = x\sum_{n=1}^{k} a_n + y\sum_{n=1}^{k} b_n.$$

The result now follows by the arithmetic properties of limits of sequences.

2. We write S_n for the partial sums of the series $\sum_{n=1}^{\infty} a_n$. We let $m \in \mathbb{N}$ with $m \geq 2$ and note that for $k \in \mathbb{N}$ with $k > m$,

$$S_k = S_{m-1} + \sum_{n=m}^{k} a_n.$$

Thus $(S_k)_{k\in\mathbb{N}}$ is convergent if and only if $\left(\sum_{n=m}^{m+k-1} a_n\right)_{k\in\mathbb{N}}$ is convergent.

3. We simply need to note that the sequence of partial sums S_n is monotone increasing (since each $a_n \geq 0$) and bounded, so convergent.

□

We now state a test for convergence or divergence, the comparison test.

Theorem 6.5. *Let* $(a_n)_{n\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$ *be sequences.*

1. *If there exists* $N \in \mathbb{N}$ *such that for all* $n \geq N$, $0 \leq a_n \leq b_n$ *and the series* $\sum_{n=1}^{\infty} b_n$ *is convergent then* $\sum_{n=1}^{\infty} a_n$ *is convergent.*

2. *If there exists* $N \in \mathbb{N}$ *such that for all* $n \geq N$, $0 \leq b_n \leq a_n$ *and the series* $\sum_{n=1}^{\infty} b_n$ *is divergent then* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

Proof. 1. By Proposition 6.4 we know that the series $\sum_{n=N}^{\infty} b_n$ is convergent. Let $b = \sum_{n=N}^{\infty} b_n$ and note that since for each $n \geq N$, $b_n \geq 0$ we must have that for all $k \in \mathbb{N}$ with $k \geq N$ we have that $\sum_{n=N}^{k} b_n \leq b$. Now for any $k \in \mathbb{N}$ with $k \geq N$ we have that

$$\sum_{n=N}^{k} a_n \leq \sum_{n=N}^{k} b_n \leq b$$

and so by the final part of Proposition 6.4 we know that $\sum_{n=N}^{\infty} a_n$ is convergent and so by the second part of Proposition 6.4 the series $\sum_{n=1}^{\infty} a_n$ is convergent.

2. By Proposition 6.4 we know that the series $\sum_{n=N}^{\infty} b_n$ is divergent. Since the sequence of partial sums $(\sum_{n=N}^{k} b_n)_{(k\geq N)}$ is increasing and divergent it must be unbounded above. So for any $x \in \mathbb{R}$ there exists $k \in \mathbb{N}$ such that $\sum_{n=N}^{k} b_n \geq x$. Thus

$$\sum_{n=N}^{k} a_n \geq \sum_{n=N}^{k} b_n \geq x.$$

Therefore the series $\sum_{n=N}^{k} a_n$ is divergent and thus by Proposition 6.4 we know that $\sum_{n=1}^{\infty} a_n$ is divergent.

□

We now give a couple of examples of using the comparison test.

Examples. *Consider the series* $\sum_{n=1}^{\infty} \frac{1}{n^2}$. *Now for all* $n \in \mathbb{N}$ *we have that* $0 \leq \frac{1}{n^2} \leq \frac{2}{n(n+1)}$. *We have already shown that* $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ *is convergent and so the series* $\sum_{n=1}^{\infty} \frac{2}{n(n+1)}$ *is convergent. Thus by the comparison test* $\sum_{n=1}^{\infty} \frac{1}{n^2}$ *is convergent.*

On the other hand consider the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$. *We know that for all* $n \in \mathbb{N}$, $0 \leq \frac{1}{n} \leq \frac{1}{\sqrt{n}}$ *and since we already know that the series* $\sum_{n=1}^{\infty} \frac{1}{n}$ *is divergent we know that the series* $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ *is divergent.*

A downside of the comparison test is you need to find a series which you already know to compare with. However if we know the behaviour of large classes of series it can be a powerful tool. We now give a couple of families where we can always determine whether the series is divergent or convergent. The first family is the geometric series.

Proposition 6.6. *Let* $a, r \in \mathbb{R}$ *with* $a \neq 0$. *We have that the series* $\sum_{n=1}^{\infty} ar^{n-1}$ *is convergent if and only if* $|r| < 1$.

Proof. If $|r| \geq 1$ then the sequence (ar^{n-1}) is divergent and so the series $\sum_{n=1}^{\infty} ar^{n-1}$ is divergent. On the other hand if $|r| < 1$ then the series of partial sums S_n satisfies that $S_n = \frac{a(1-r^n)}{1-r}$ and so $\lim_{n\to\infty} S_n = \frac{a}{1-r}$ which means that $\sum_{n=1}^{\infty} ar^{n-1}$ is convergent. □

The second family includes the Harmonic series

Proposition 6.7. *Let* $\alpha \in \mathbb{Q}$. *The series* $\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$ *is convergent if and only if* $\alpha > 1$.

Proof. If $\alpha \leq 1$ then $n^{-\alpha} \geq n^{-1}$ for all $n \in \mathbb{N}$. Therefore since the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, by the comparison test the series $\sum_{n=1}^{\infty} n^{-\alpha}$ is divergent.

On the other hand suppose that $\alpha > 1$. We have that for all $n \in \mathbb{N}$,

$$S_{2^{n+1}-1} - S_{2^n-1} = \sum_{i=2^n}^{2^{n+1}-1} i^{-\alpha} \leq \sum_{i=2^n}^{2^{n+1}-1} 2^{-n\alpha} = 2^n 2^{-n\alpha} = 2^{n(1-\alpha)}.$$

Thus if $2^n \leq k \leq 2^{n+1} - 1$ we have that

$$S_k \leq S_{2^{n+1}-1} = S_1 + \sum_{j=1}^{n} (S_{2^{j+1}-1} - S_{2^j-1}) \leq 1 + \sum_{j=1}^{n} 2^{j(1-\alpha)} \leq 1 + \frac{2^{1-\alpha}}{1 - 2^{1-\alpha}}.$$

So the sequence of partial sums S_n is bounded and so by part 4 of Proposition 6.4 we have the series $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ is convergent. $\square$

Remark. *We include the assumption that $\alpha \in \mathbb{Q}$ since n^α since it has not been made clear yet how to define n^α if $\alpha \notin \mathbb{Q}$. However once you see the definition of n^α for any $\alpha \in \mathbb{R}$ (later on in analysis 1B or see page 170 of Howie) it can be shown that the result is true for any $\alpha \in \mathbb{R}$.*

We can now use these results to apply the comparison test.

Examples. *Let $a_n = \frac{n^2+4n+5}{n^4+5n+3}$ for all $n \in \mathbb{N}$. We have that*

$$a_n = \frac{n^2}{n^4} \left(\frac{1 + 4n^{-1} + 5n^{-2}}{1 + 5n^{-3} + 3n^{-4}} \right).$$

Since $\lim_{n\to\infty} \left(\frac{1+4n^{-1}+5n^{-2}}{1+5n^{-3}+3n^{-4}} \right) = 1$. We can find $N \in \mathbb{N}$ such that for all $n \geq N$,

$$0 \leq \left(\frac{1 + 4n^{-1} + 5n^{-2}}{1 + 5n^{-3} + 3n^{-4}} \right) \leq \frac{3}{2}.$$

Thus for all $n \geq N$, $0 \leq a_n \leq \frac{3}{2n^2}$ and since $\sum_{n=1}^{\infty} n^{-2}$ is convergent we can conclude by the comparison test and part 1 of Proposition 6.4 that $\sum_{n=1}^{\infty} a_n$ is convergent. This method can be adapted to any series of a similar form (find the dominate term in the numerator and denominator and take these factors out).

For a simpler example let $b_n = \frac{n}{n^2+1}$. Then for all $n \in \mathbb{N}$ $b_n \geq \frac{1}{2n} \geq 0$ and since $\sum_{n=1}^{\infty} n^{-1}$ is divergent by the comparison test $\sum_{n=1}^{\infty} b_n$ divergent.

We can combine what we know about the geometric series and the comparison test to give the following convergence test

Theorem 6.8 (Ratio test)**.** *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers where $\left(\frac{a_{n+1}}{a_n}\right)_{n\in\mathbb{N}}$ is convergent.*

1. *If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$ then the series $\sum_{n=1}^{\infty} a_n$ is convergent.*

2. *If* $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$ *then the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

3. *If* $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ *then the series* $\sum_{n=1}^{\infty} a_n$ *may be convergent or divergent.*

Proof. 1. Suppose that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$ and let $1 > r > \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$. Thus there exists N such that for all $n \geq N$ $\frac{a_{n+1}}{a_n} \leq r$. We show by induction that for all $k \geq N$, $a_k \leq r^{k-N} a_N$. It is clearly true if $k = N$. If we suppose it is true for some $j \geq N$ then $a_{j+1} \leq r a_j \leq r r^{j-k} a_N = r^{(j+1)-k} a_N$. Thus since the geometric series $\sum_{k=N}^{\infty} r^{k-N} a_N$ is convergent by the comparison test $\sum_{k=N}^{\infty} a_k$ is convergent and so by Proposition 6.4 $\sum_{n=1}^{\infty} a_n$ is convergent.

2. Suppose that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$ and let $1 < r < \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$. Thus there exists N such that for all $n \geq N$ $\frac{a_{n+1}}{a_n} \geq r$. So again we can easily show by induction that for all $n \geq N$, $a_k \geq r^{k-N} a_N$. Since the geometric series $\sum_{k=N}^{\infty} r^{k-N} a_N$ is divergent it follows by the comparison test and Proposition 6.4 that $\sum_{k=1}^{\infty} a_k$ is divergent.

3. If we let $a_n = 1$ for all $n \in \mathbb{N}$ then the series $\sum_{n=1}^{\infty} a_n$ is divergent but $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$. On the other hand if $a_n = n^{-2}$ for all $n \in \mathbb{N}$ then $\frac{a_{n+1}}{a_n} = \frac{(n+1)^{-2}}{n^{-2}} = \frac{n^2}{(n+1)^2}$ and $\lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1$ but $\sum_{n=1}^{\infty} n^{-2}$ is convergent.

□

In the case where $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ or if the sequence $(\frac{a_{n+1}}{a_n})$ is not convergent we say that the ratio test is inconclusive (in the second case there are sometimes alternative versions which would work).

The test below is a related test which is sometimes more convenient to apply.

Theorem 6.9 (Root test). *Let* $(a_n)_{n\in\mathbb{N}}$ *be a sequence of positive real numbers where* $((a_n)^{1/n})$ *is convergent.*

1. *If* $\lim_{n\to\infty} a_n^{1/n} < 1$ *then the series* $\sum_{n=1}^{\infty} a_n$ *is convergent.*

2. *If* $\lim_{n\to\infty} a_n^{1/n} > 1$ *then the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

3. *If* $\lim_{n\to\infty} a_n^{1/n} = 1$ *then the series* $\sum_{n=1}^{\infty} a_n$ *may be convergent or divergent.*

Proof. See exercise sheet 7. □

Remark. *In fact it is possible to show that for a sequence of positive real numbers* $(a_n)_{n\in\mathbb{N}}$ *if* $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r$ *then* $\lim_{n\to\infty} a_n^{1/n} = r$ *and so the root test implies the ratio test. However there are cases where the root test can be applied but the ratio test cannot be applied see the additional problems on series.*

Example. *Let* $a_n = \frac{5n}{2^n}$. *We have that*

$$\frac{a_{n+1}}{a_n} = \frac{\frac{5(n+1)}{2^{n+1}}}{\frac{5n}{2^n}} = \frac{5(n+1)}{10n}.$$

Thus $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{2}$ *and since* $a_n \geq 0$ *for all* $n \in \mathbb{N}$, *by the ratio test the series* $\sum_{n=1}^{\infty} a_n$ *is convergent.*

Here is a summary of what to consider if you are asked whether $\sum_{n=1}^{\infty} a_n$ is convergent or divergent.

1. Check that $\lim_{n\to\infty} a_n = 0$. If this is not the case the series is divergent.

2. Can you compute the sequence of partial sums S_n directly? If you can see whether or not the sequence $(S_n)_{n\in\mathbb{N}}$ is convergent or divergent.

3. Check whether you can apply the comparison test.

4. Check whether you can apply the ratio or root test.

This completes the section on series for analysis 1a. In analysis 1b you will see more series in particular more attention will be paid to the case when a_n takes both positive and negative values.

7 Functions

We now move to making use of our work on sequences to study the behaviour of real valued functions. We will be considering functions $f : A \to \mathbb{R}$ where $A \subseteq \mathbb{R}$ is non-empty. For a formal introduction to functions see section 1.2 of `https://people.maths.bris.ac.uk/~sz16567/F&P%20-%20Section%201.pdf`
We call A the domain of the function, $\mathbb{R}$ the codomain and the range is the set

$$f(A) = \{f(x) : x \in A\}.$$

If we have two functions $f, g \in A \to \mathbb{R}$ then we define functions $f+g, f\cdot g, |f| : A \to \mathbb{R}$ by

$$(f+g)(x) = f(x)+g(x),\ (f\cdot g)(x) = f(x)g(x) \text{ and } |f|(x) = |f(x)| \text{ for all } x \in A.$$

The first concept we want to look at is when a function is bounded

Definition 7.1. *Let $A \subseteq \mathbb{R}$ be non-empty and $f : A \to \mathbb{R}$ a function. We say that f is bounded above if there exists $x \in \mathbb{R}$ such that for all $a \in A$, $f(a) \leq x$. We say that f is bounded below if there exists $x \in \mathbb{R}$ such that for all $a \in A$, $f(a) \geq x$. We say that f is bounded if it is both bounded above and bounded below.*

Example. *The function $f : (0, 1) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is unbounded. To see this let $y \in \mathbb{R}$ with $y \geq 1$. We then have that $0 < \frac{1}{2y} < 1$ and $f(1/2y) = 2y > y$ and so f is not bounded above and so is unbounded.*

Example. *The function $f : [1, \infty) \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is bounded. For all $x \in [1, \infty)$ we have that $0 \leq f(x) = \frac{1}{x} \leq 1$.*

We now turn our attention to limits of functions. First of all we consider limits as x goes to infinity. Note that for this notion to make sense we need to make sure our domain A is unbounded above. If $A = (0, 1)$ and $f : (0, 1) \to \mathbb{R}$ then it makes no sense to talk about limits as x goes to infinity.

Definition 7.2. *Let $A \subset \mathbb{R}$ be unbounded above and $f : A \to \mathbb{R}$ be a function. We make the following definition*

1. *We say that $\lim_{x\to\infty} f(x) = y$ where $y \in \mathbb{R}$ if and only if for all $\epsilon > 0$ there exists $r \in \mathbb{R}$ such that if $x \in A$ and $x \geq r$ then $|f(x) - y| \leq \epsilon$.*
2. *We say that $\lim_{x\to\infty} f(x) = \infty$ if for all $z \in \mathbb{R}$ there exists $r \in \mathbb{R}$ such that if $x \in A$ and $x \geq r$ then $f(x) \geq z$.*
3. *We say that $\lim_{x\to\infty} f(x) = -\infty$ if for all $z \in \mathbb{R}$ there exists $r \in \mathbb{R}$ such that if $x \in A$ and $x \geq r$ then $f(x) \leq z$.*

Note that we can make analogous statements about the limit as x goes to minus infinity.

Examples. *Define* $f : (0, \infty) \to \mathbb{R}$ *by* $f(x) = \frac{x+1}{x}$. *We have that* $\lim_{x\to\infty} f(x) = 1$. *Let* $\epsilon > 0$, *choose* $r = \epsilon^{-1}$ *then if* $x \geq r$ *then*

$$|f(x) - 1| = \left|\frac{x+1}{x} - 1\right| = \left|\frac{x+1-x}{x}\right| = \frac{1}{x} \leq \frac{1}{r} = \epsilon.$$

Define $g : (0, \infty) \to \mathbb{R}$ *by* $g(x) = \frac{x^2+1}{x}$. *We have that* $\lim_{x\to\infty} g(x) = \infty$. *Let* $z \in \mathbb{R}$, *choose* $r = z$ *then if* $x \geq r$ *we have that*

$$f(x) = \frac{x^2+1}{x} \geq \frac{x^2}{x} = x \geq r \geq z.$$

Finally if we define $h : \mathbb{R} \to \mathbb{R}$ *by* $h(x) = \sin(x)$ *then* h *does not have a limit as* x *goes to infinity. To see this first note that* $|h(x)| \leq 1$ *for all* $x \in \mathbb{R}$ *so* h *cannot diverge to* $\pm\infty$ *as* x *goes to infinity. Now suppose that* $\lim_{x\to\infty} h(x) = \alpha$ *for some* $\alpha \in \mathbb{R}$. *Thus there exists* $z \in \mathbb{R}$ *such that for all* $x \geq z$, $|\sin x - \alpha| \leq 1/4$. *Now by the Archimedian principle we can find* $n \in \mathbb{N}$ *with* $n\pi > z$ *and since* $\sin(n\pi) = 0$ *we must have that* $|\alpha| \leq \frac{1}{4}$. *However* $|\sin((n+1/2)\pi)| = 1$ *and so* $|\alpha| \geq 3/4$ *and we have a contradiction.*

Similarly to the case for sequences we can prove results about the arithmetic properties of limits. The most convenient way of doing this is to prove the following statement relating limits of functions to limits of sequences.

Proposition 7.3. *Let* $A \subseteq \mathbb{R}$ *be unbounded above,* $f : A \to \mathbb{R}$ *be a function and* $y \in \mathbb{R}$. *We have that* $\lim_{x\to\infty} f(x) = y$ *if and only if for all sequences* $(x_n)_{n\in\mathbb{N}}$ *where* $x_n \in A$ *for all* $n \in \mathbb{N}$ *and* $\lim_{n\to\infty} x_n = \infty$ *we have that* $\lim_{n\to\infty} f(x_n) = y$.

Proof. Suppose that $\lim_{x\to\infty} f(x) = y$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence where $x_n \in A$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = \infty$. We need to show $\lim_{n\to\infty} f(x_n) = y$. Let $\epsilon > 0$ and note that since $\lim_{x\to\infty} f(x) = y$ we can choose $r \in \mathbb{R}$ such that if $x \geq r$ then $|f(x) - y| \leq \epsilon$. Since $\lim_{n\to\infty} x_n = \infty$ we can choose N such that if $n \geq N$ then $x_n \geq r$. Thus for all $n \geq N$ we have that $x_n \geq r$ and so $|f(x_n) - y| \leq \epsilon$. So we have shown that $\lim_{n\to\infty} f(x_n) = y$.

We also need to show that if for all sequences $(x_n)_{n\in\mathbb{N}}$ where $x_n \in A$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = \infty$ we have that $\lim_{n\to\infty} f(x_n) = y$ then $\lim_{x\to\infty} f(x) = y$. This is equivalent to showing that if $\lim_{n\to\infty} f(x) \neq y$ then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ where $x_n \in A$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = \infty$ but for which $\lim_{n\to\infty} f(x_n) \neq y$. Since $\lim_{n\to\infty} f(x) \neq y$ there exists $\epsilon > 0$ such that for all $r \in \mathbb{R}$ there exists $x \in A$ with $x \geq r$ and $|f(x) - y| > \epsilon$. Thus we can find $x_1 \in A$ with $x_1 \geq 1$ and $|f(x_1) - y| > \epsilon$. Similarly for all $k \in \mathbb{N}$ we can find $x_k \in A$ with $x_k \geq k$ and $|f(x_k) - y| > \epsilon$. Thus we have a sequence $(x_n)_{n\in\mathbb{N}}$ where $x_n \in A$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = \infty$ but for which $\lim_{n\to\infty} f(x_n) \neq y$. □

Thus we can deduce behaviour of limits for sums, products and quotients of functions using the results we already have for sequences.

Theorem 7.4. *Let $A \subseteq \mathbb{R}$ be a set which is not bounded above. Let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ be functions and $a, b \in \mathbb{R}$ where $\lim_{x\to\infty} f(x) = a$ and $\lim_{x\to\infty} g(x) = b$. We have that*

1. $\lim_{x\to\infty} f(x) + g(x) = a + b$
2. $\lim_{x\to\infty} f(x)g(x) = ab$
3. *If $g(x) \neq 0$ for any $x \in A$ and $b \neq 0$ then* $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{a}{b}$.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence such that each $x_n \in A$ and for which $\lim_{n\to\infty} x_n = \infty$. We have that $\lim_{n\to\infty} f(x_n) = a$ and $\lim_{n\to\infty} g(x_n) = b$. Therefore by the arithmetic properties of limits of sequences

1. $\lim_{n\to\infty} f(x_n) + g(x_n) = a + b$
2. $\lim_{n\to\infty} f(x_n)g(x_n) = ab$
3. If $g(x) \neq 0$ for all $x \in A$ and $b \neq 0$ then $\lim_{n\to\infty} \frac{f(x_n)}{g(x_n)} = \frac{a}{b}$.

Thus the result follows by Proposition 7.3. □

So far this is very similar to the case for sequences. However for functions we can also think about the limit as x goes to point $a \in \mathbb{R}$ as long as a is chosen reasonably. If f has domain $(0, \infty)$ it would make no sense to talk about the limit of f as x approaches -1 but it could make sense to talk about the limit as x approaches 0. The following definition allows us to decide which values of a to consider.

Definition 7.5. *Let $A \subseteq \mathbb{R}$. We say that $x \in \mathbb{R}$ is an accumulation point of A if there exists a sequence (x_n) such that $x_n \in A$ for all $n \in \mathbb{N}$, $\lim_{n\to\infty} x_n = x$ and $x_n \neq x$ for any $n \in \mathbb{N}$.*

Examples. 1 *is an accumulation point of the set* $(0, 1)$ *despite not being in the set as if we let $x_n = 1 - \frac{1}{2n}$ then $\lim_{n\to\infty} x_n = 1$ and each $x_n \in (0, 1)$.*

However 2 is not an accumulation point of the set $(0, 1) \cup \{2\}$ since the only way we can find $(x_n)_{n\in\mathbb{N}}$ with each $x_n \in (0, 1) \cup \{2\}$ and $\lim_{n\to\infty} x_n = 2$ is if $x_n = 2$ for all n sufficiently large.

We can now introduce the notion of the limit as f goes to a real value a.

Definition 7.6. *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ a function and a be an accumulation point of A. We make the following definitions.*

1. *We say that $\lim_{x\to a} f(x) = y$ where $y \in \mathbb{R}$ if and only if for all $\epsilon > 0$ there exists $\delta > 0$ such that if $x \in A$ and $0 < |x - a| \leq \delta$ then $|f(x) - y| \leq \epsilon$.*

2. *We say that* $\lim_{x\to a} f(x) = \infty$ *if and only if for all* $r \in \mathbb{R}$ *there exists* $\delta > 0$ *such that if* $x \in A$ *and* $0 < |x - a| \leq \delta$ *then* $f(x) \geq r$.

3. *We say that* $\lim_{x\to a} f(x) = -\infty$ *if and only if for all* $r \in \mathbb{R}$ *there exists* $\delta > 0$ *such that if* $x \in A$ *and* $0 < |x - a| \leq \delta$ *then* $f(x) \leq r$.

Examples. *Define* $f_1 : (0,1) \to \mathbb{R}$ *by* $f(x) = \frac{x-1}{x}$. *We have that* $\lim_{x\to 0} f_1(x) = -\infty$ *and* $\lim_{x\to 1} f_1(x) = 0$. *To see that* $\lim_{x\to 1} f(x) = 0$ *we let* $\epsilon > 0$ *and choose* $\delta = \min\{\epsilon/2, 1/2\}$. *If* $x \in (0,1)$ *and* $|x - 1| \leq \delta$ *then* $|x| \geq 1/2$. *Therefore*

$$\left|\frac{x-1}{x}\right| = \frac{|x-1|}{|x|} \leq \frac{\delta}{|x|} \leq 2\delta \leq \epsilon.$$

To show that $\lim_{x\to 0} f_1(x) = -\infty$. *Let* $y \in \mathbb{R}$ *and note that if* $y > 0$ *then* $f(x) \leq y$ *for all* $x \in (0,1)$ *so we may assume that* $y \leq 0$. *In this case choose* $\delta = \frac{1}{1-y}$ *and we have that if* $x < \delta$ *then* $x \in (0, \delta)$ *and*

$$f_1(x) = \frac{x}{1-x} = 1 - \frac{1}{x} \leq 1 - \frac{1}{\frac{1}{1-y}} = y.$$

Now let $f_2 : (1, \infty) \to \mathbb{R}$ *be defined by* $f_2(x) = \frac{(x-1)}{(x^2-1)}$. *For* $x > 1$ *we have that* $f_2(x) = \frac{1}{(x+1)}$ *and we will use this to show that* $\lim_{x\to 1} f_2(x) = \frac{1}{2}$. *We let* $\epsilon > 0$ *and choose* $\delta = 4\epsilon$. *If* $|x - 1| \leq \delta$ *then* $x \in (1, 1 + \delta]$. *So*

$$\left|\frac{1}{x+1} - \frac{1}{2}\right| = \left|\frac{2-(x+1)}{2(x+1)}\right| = \frac{x-1}{2(x+1)} \leq \frac{\delta}{4} = \epsilon.$$

Let $f_3 : \mathbb{R} \to \mathbb{R}$ *be defined by*

$$f_3(x) = \begin{cases} 0 & \textit{if} \quad x \neq 0 \\ 1 & \textit{if} \quad x = 0 \end{cases}$$

We have that $\lim_{x\to 0} f_3(x) = 0$ *despite the fact that* $f_3(x) \neq 0$. *To see this let* $\epsilon > 0$ *and choose* $\delta > 0$ *for any* $x \in \mathbb{R}$ *with* $0 < |x| < \delta$ *(note that we cannot have* $x = 0$ *here which is crucial for this example) we have that* $|f(x) - 0| = 0 \leq \epsilon$.

Finally let Let $f_4 : \mathbb{R} \to \mathbb{R}$ *be defined by*

$$f_4(x) = \begin{cases} 0 & \textit{if} \quad x \leq 0 \\ 1 & \textit{if} \quad x > 0 \end{cases}$$

f_4 *does not have a limit as* x *approaches* 0. *To see this let* $\alpha \in \mathbb{R}$, *choose* $\epsilon = \frac{1}{4}$ *and let* $\delta > 0$. *We have that* $|\delta - 0| \leq \delta$ *and*

$$|f_4(\delta) - f_4(-\delta)| = 1.$$

However by the triangle inequality

$$1 \leq |f_4(\delta) - \alpha| + |f_4(-\delta) - \alpha|$$

and so one of $|f_4(\delta) - \alpha|$ *or* $|f_4(-\delta) - \alpha|$ *must be greater than* ϵ. *Thus no limit as* x *goes to* 0 *exists.*

In fact there is a convenient alternative definition similar to the case for the limit when x goes to infinity.

Theorem 7.7 (Heine definition)**.** *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, $a \in \mathbb{R}$ be an accumulation point of A and $y \in \mathbb{R}$. We have that $\lim_{x\to a} f(x) = y$ if and only if for all sequences $(x_n)_{n\in\mathbb{N}}$ where $x_n \in A\backslash\{a\}$ for each $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = a$ we have that $\lim_{n\to\infty} f(x_n) = y$.*

Proof. First of all we suppose that $\lim_{x\to a} f(x) = y$. We then let $(x_n)_{n\in\mathbb{N}}$ be a sequence such that $x_n \in A\backslash\{a\}$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = a$. Let $\epsilon > 0$. Since $\lim_{x\to a} f(x) = y$ we can find $\delta > 0$ such that if $x \in A$ and $0 < |x - a| \leq \delta$ then $|f(x) - f(a)| \leq \epsilon$. Since $\lim_{n\to\infty} x_n = a$ we can find $N \in \mathbb{N}$ such that if $n \geq N$ then $0 < |x_n - a| \leq \delta$ and thus $|f(x_n) - y| \leq \epsilon$.

For the other direction we will suppose that $\lim_{x\to a} f(x) \neq y$ and show this means we can find a sequence $(x_n)_{n\in\mathbb{N}}$ where each $x_n \in A$, $\lim_{n\to\infty} x_n = a$ but $\lim_{n\to\infty} f(x) \neq y$. Suppose that $\lim_{x\to a} f(x) \neq y$. Thus there exists $\epsilon > 0$ such that for all $\delta > 0$ there exists $x \in A$ with $0 < |x - a| \leq \delta$ but $|f(x) - f(a)| > \epsilon$. So we fix such an ϵ and note that for all $n \in \mathbb{N}$ by taking $\delta = \frac{1}{n}$ we can find $x_n \in A$ such that $0 < |x_n - a| \leq \frac{1}{n}$ and $|f(x_n) - y| > \epsilon$. Therefore $\lim_{n\to\infty} x_n = a$ but $f(x_n)$ does not converge to y.

□

Example. *As above let $g : (1, \infty) \to \mathbb{R}$ be defined by $g(x) = \frac{(x-1)}{(x^2-1)}$. For $x > 1$ we have that $g(x) = \frac{1}{(x+1)}$ we will show that $\lim_{x\to 1} g(x) = \frac{1}{2}$. Let (x_n) be a sequence where each $(x_n) \in (0, \infty)$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = 1$. We have that $f(x_n) = \frac{1}{x_n+1}$ since $x_n + 1 \neq 0$ for any $n \in \mathbb{N}$ we have by the arithmetic properties of limits that $\lim_{n\to\infty} f(x_n) = \frac{1}{2}$.*

Using this result we can use what we already know about limits to prove the following.

Theorem 7.8. *Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$ be a function, a be an accumulation point of A and $b, c \in \mathbb{R}$. If $\lim_{x\to a} f(x) = b$ and $\lim_{x\to a} g(y) = c$ then*

1. $\lim_{x\to a}(f + g)(x) = b + c$,
2. $\lim_{x\to a}(f \cdot g)(x) = bc$,
3. *if $g(x) \neq 0$ for all $x \in A$ and $c \neq 0$ then $\lim_{x\to a} \frac{f}{g}(x) = \frac{b}{c}$*

Proof. This is left as an exercise (use the Heine definition and then you can exploit the results you already have for sequences). □

7.1 Continuity

We now turn to the notion of when a function is continuous. Continuity intuitively means that a function does not suddenly jump i.e we can draw a graph of the function without removing pen from paper. However we need a formal definition.

Definition 7.9. *Let* $A \subseteq \mathbb{R}$ *and* $f : A \to \mathbb{R}$ *be a function. We say that* f *is continuous at a point* $a \in A$ *if and only if for all* $\epsilon > 0$ *there exists* $\delta > 0$ *such that if* $x \in A$ *and* $|x - a| \leq \delta$ *then* $|f(x) - f(a)| \leq \epsilon$*.*

We can also restate this definition using limits of functions or sequences.

Definition 7.10. *Let* $A \subseteq \mathbb{R}$ *and* $f : A \to \mathbb{R}$ *be a function. We say that* f *is continuous at a point* $a \in A$ *if and only if either* a *is not an accumulation point of* A *or if* a *is an accumulation point of* A $\lim_{x\to a} f(x) = f(a)$*.*

To see these two definitions are equivalent first of all note that if a is not an accumulation point of A then there will exist $\delta > 0$ such that if $x \in A$ and $|x - a| \leq \delta$ then $x = a$ and so $|f(x) - f(a)| = 0$. If a is an accumulation point of A then we can immediately see the two definitions are identical. Finally using the Heine definition for limits of functions we can come up with the following definition

Definition 7.11. *[Heine definition] Let* $A \subseteq \mathbb{R}$ *and* $f : A \to \mathbb{R}$ *be a function. We say that* f *is continuous at a point* $a \in A$ *if and only if for all sequences* $(x_n)_{n\in\mathbb{N}}$ *where each* $x_n \in A$ *and* $\lim_{n\to\infty} x_n = a$ *we have that* $\lim_{n\to\infty} f(x_n) = f(a)$*.*

Of these definition we will mainly use the Heine definition as it allows us to use the results we already have got for sequences. However in analysis 1b you will see the notion of uniform continuity and this will require you to know the first ϵ, δ-definition.

Examples. *Let* $c \in \mathbb{R}$ *and* $f : \mathbb{R} \to \mathbb{R}$ *be the function where* $f(x) = c$ *for all* $x \in \mathbb{R}$*.* f *is continuous for all* $x \in \mathbb{R}$*. Let* $x \in \mathbb{R}$ *and* (x_n) *be a real valued sequence with* $\lim_{n\to\infty} x_n = x$*. We then have that* $f(x_n) = c$ *for all* $n \in \mathbb{N}$ *and so* $\lim_{n\to\infty} f(x_n) = c = f(x)$*.*

Let $g : \mathbb{R} \to \mathbb{R}$ *be the function defined by* $g(x) = x$ *for all* $x \in \mathbb{R}$*.* g *is continuous for all* $x \in \mathbb{R}$*. Let* $x \in \mathbb{R}$ *and* (x_n) *be a real valued sequence with* $\lim_{n\to\infty} x_n = x$*. We then have that* $g(x_n) = x_n$ *for all* $n \in \mathbb{N}$ *and so* $\lim_{n\to\infty} g(x_n) = x = g(x)$*.*

We now give an example of a function which is discontinuous at a point.

Example. *Let* $h : \mathbb{R} \to \mathbb{R}$ *be defined by*

$$h(x) = \begin{cases} 1 & \text{if} \quad x \leq 0 \\ 0 & \text{if} \quad x > 0 \end{cases}$$

h is discontinuous at 0*. We have that* $h(1) = 0$ *but if we let* $x_n = \frac{1}{n}$ *for all* $n \in \mathbb{N}$ *then* $\lim_{n\to\infty} x_n = 0$ *and* $\lim_{n\to\infty} h(x_n) = 0 \neq h(0)$ *and so* h *is discontinuous at* 0*.*

Definition 7.12. *Let* $A \subseteq \mathbb{R}$ *and* $f : A \to \mathbb{R}$ *be a function. We say that* F *is continuous on* A *or that the function* $f : A \to \mathbb{R}$ *is continuous if* f *is continuous for all* $a \in A$*.*

Thus the two functions above, f and g, are continuous on $\mathbb{R}$. This may seem trivial but by combining it with the theorem below we can show the continuity of a wide class of functions.

Theorem 7.13. *Let* $A \subseteq \mathbb{R}$ *and* $f, g : A \to \mathbb{R}$ *be continuous functions on* A*. We have that*

1. $f + g : A \to \mathbb{R}$ *is continuous on* A*.*
2. $f \cdot g : A \to \mathbb{R}$ *is continuous on* A*.*
3. *If* $g(x) \neq 0$ *for all* $\in A$ *and we define* $h : A \to \mathbb{R}$ *by* $h(a) = \frac{f(a)}{g(a)}$ *then* h *is continuous on* A*.*

Proof. This is an exercise. Let $a \in A$ and use the Heine definition of continuity combined with the results we already have for sequences. $\square$

Definition 7.14. *Let* $f, g : \mathbb{R} \to \mathbb{R}$ *be polynomials (i.e.* $f(x) = \sum_{n=0}^{k_1} a_n x^n$ *and* $g(x) = \sum_{n=0}^{k_2} b_n x^n$ *where each* $a_i \in \mathbb{R}$ *and each* $b_i \in \mathbb{R}$*). We define*

$$Z(g) = \{x \in \mathbb{R} : g(x) = 0\}.$$

We can define a function $h : \mathbb{R}\backslash Z(g) \to \mathbb{R}$ *by* $h(x) = \frac{f(x)}{g(x)}$*.* h *is called a rational function.*

Corollary 7.15. *A rational function* $h = f/g$ *is continuous on* $\mathbb{R}\backslash Z(g)$*.*

Proof. Let $f, g : \mathbb{R} \to \mathbb{R}$ be polynomials and $h : \mathbb{R}\backslash Z(g) \to \mathbb{R}$ be the rational function defined by $h(x) = \frac{f(x)}{g(x)}$. We know that since constant functions are continuous and since the function $t : \mathbb{R} \to \mathbb{R}$ defined by $t(x) = x$ that both the functions f and g will be continuous by Theorem 7.13. Now let $a \in \mathbb{R}\backslash Z(g)$ and (x_n) a sequence where each term $x_n \in \mathbb{R}\backslash Z(g)$ and $\lim_{n\to\infty} x_n = a$. We know that $\lim_{n\to\infty} g(x_n) = g(a)$ and $\lim_{n\to\infty} f(x_n) = h(a)$. Since $g(x_n) \neq 0$ for all $n \in \mathbb{N}$ and $g(a) \neq 0$ by the quotient rule for sequences

$$\lim_{n\to\infty} \frac{f(x_n)}{g(x_n)} = \frac{f(a)}{g(a)} = h(a).$$

Thus h is continuous for all $a \in \mathbb{R}\backslash Z(g)$. $\square$

The following result gives a simple way of constructing discontinuous functions.

Lemma 7.16. *1. For all $x \in \mathbb{Q}$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} x_n = x$ and each $x_n \in \mathbb{R}\backslash\mathbb{Q}$.*

2. For all $x \in \mathbb{R}\backslash\mathbb{Q}$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} x_n = x$ and each $x_n \in \mathbb{Q}$.

Proof. 1. If $x \in \mathbb{Q}$ then we can simply take the sequence $(x_n)_{n\in\mathbb{N}}$ where $x_n = x + \frac{\sqrt{2}}{n}$ and we have that each x_n is irrational and $\lim_{n\to\infty} x_n = x$.

2. If $x \notin \mathbb{Q}$ then the intuitive way of seeing this would be to write x by its infinite decimal expansion which provides a sequence converging to x. To give a formal proof we will use the Archimedean property. Let $n \in \mathbb{N}$ and note by the Archimedean principle we can find a natural number k such that $k \in (nx, nx+1]$. Thus if we let $x_n = \frac{k}{n}$ then $x_n \in (x, x+\frac{1}{n})$. Therefore (x_n) is a sequence of rational numbers which converge to x.

□

Example. *Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by*

$$f(x) = \begin{cases} 1 & \text{if} \quad x \in \mathbb{Q} \\ 0 & \text{if} \quad x \notin \mathbb{Q} \end{cases}$$

The function f is discontinuous for all $x \in \mathbb{R}$. First of all if $x \in \mathbb{Q}$ then by Lemma 7.16 we can find a sequence (x_n) such that $x_n \notin \mathbb{Q}$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = x$. So for all $n \in \mathbb{N}$ $f(x_n) = 0$ which does not converge to $f(x) = 1$. So f is discontinuous at x.

Now suppose that $x \notin \mathbb{Q}$. By Lemma 7.16 we can find a sequence (x_n) such that $x_n in \mathbb{Q}$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = x$. So $f(x) = 0$ but for all $n \in \mathbb{N}$ we have that $f(x_n) = 1$. So again $\lim_{n\to\infty} f(x_n) \neq f(x)$ so f is discontinuous at x.

The example above can be modified to give examples of functions continuous just at one point and a function continuous at all irrationals but discontinuous at every rational.

7.2 Extremal Value Theorem and Intermediate Value Theorem

We now turn to two very important results regarding continuous functions on closed intervals. Our first such result is the extremal value theorem.

Theorem 7.17 (Extremal value theorem). *Let $a < b$ be real numbers. If $f : [a, b] \to \mathbb{R}$ is continuous then f is bounded and there exist $x, y \in [a.b]$ such that*

$$f(x) = \sup\{f(c) : c \in [a, b]\}$$

and

$$f(y) = \inf\{f(c) : c \in [a, b]\}.$$

Proof. We will start by showing that f is bounded above by contradiction. We suppose that f is not bounded above and thus we can find a sequence $(x_n)_{n\in\mathbb{N}}$ where each $x_n \in [a, b]$ but $\lim_{n\to\infty} f(x_n) = \infty$. Since (x_n) is a bounded sequence by the Bolzano-Weierstrass Theorem we can find a subsequence of (x_n), $(x_{n_k})_{k\in\mathbb{N}}$ which is convergent. We will let $x = \lim_{k\to\infty} x_{n_k}$. Since f is continuous we must have that $f(x) = \lim_{k\to\infty} f(x_{n_k})$ and since $f(x) \in \mathbb{R}$ this means that $(f(x_{n_k}))$ is convergent but this contradicts the fact that $\lim_{n\to\infty} f(x_n) = \infty$.

We now show that there exists $x \in [a, b]$ with

$$f(x) = \sup\{f(c) : c \in [a, b]\}.$$

We let $z = \sup\{f(c) : c \in [a, b]\}$ and find a sequence $(x_n)_{n\in\mathbb{N}}$ where $\lim_{n\to\infty} f(x_n) = z$ and for all $n \in \mathbb{N}$, $x_n \in [a, b]$ (why is this possible?). By the Bolzano-Weierstrass Theorem we can find a subsequence (x_{n_k}) of (x_n) which is convergent. We let $x = \lim_{k\to\infty} x_{n_k}$ and note that $x \in [a, b]$. Thus by the continuity of f we have that

$$f(x) = \lim_{k\to\infty} f(x_{n_k}) = \lim_{n\to\infty} f(x_n) = z$$

.

To prove that f is bounded below and achieves its infimum we simply need to apply the above argument to $-f$. □

If we have a continuous function with domain an open interval (a, b) or (a, ∞) then the result does not necessarily hold as the following examples show.

Examples.

1. *Let $f : (0, 1) \to \mathbb{R}$ be defined by $f(x) = \frac{1}{x}$ then f is continuous but on $(0, 1)$ but not bounded.*
2. *Let $g : (0, 1) \to \mathbb{R}$ be defined by $g(x) = x$. Then g is continuous and bounded but there does not exist $x \in (0, 1)$ with*

   $$g(x) = \sup\{g(y) : y \in (0, 1)\} = 1$$

3. *Let $h : (0, \infty) \to \mathbb{R}$ be defined by $h(x) = \frac{x-1}{x}$. h is continuous but not bounded below. Furthermore h is bounded above with*

   $$\sup\{h(x) : x \in (0, \infty)\} = 1.$$

   However there does not exist $x \in (0, \infty)$ with $h(x) = 1$.

However if we are given more information about the function $f : (a, b) \to \mathbb{R}$ sometimes we can get round these problems.

Example. *Let $f : (0, 1] \to \mathbb{R}$ satisfy that $f(1) = 1$ and $\lim_{x\to 0} f(x) = 0$. We will show that there exists $x \in (0, 1]$ with*

$$f(x) = \sup\{f(c) : c \in (0, 1]\}.$$

Since $\lim_{x\to 0} f(x) = 0$ we can find $\delta > 0$ such that if $|x| \leq \delta$ then $|f(x)| \leq \frac{1}{2}$. Thus for all $y \in [0, \delta]$ we have that $f(y) < 1$. Now we can apply the extremal value theorem to f where we restrict the domain to $[\delta, 1]$. Thus there exists $x \in [\delta, 1]$ such that

$$f(x) = \sup\{f(c) : c \in [\delta, 1]\} \geq f(1) = 1.$$

Furthermore if $0 \leq y \leq \delta$ then $f(x) \geq 1 > \frac{1}{2} \geq f(y)$. So

$$f(x) = \sup\{f(c) : c \in (0, 1]\} \geq 1.$$

We now turn to the intermediate value theorem which is the final result of the unit and one of the most important. It is a result you will have already used several times before probably without realising it.

Theorem 7.18. *[Intermediate Value Theorem] Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ be a continuous function where $f(a) < f(b)$. We have that for all $y \in (f(a), f(b))$ there exists $x \in (a, b)$ such that $f(x) = y$.*

Proof. We fix $y \in (f(a), f(b))$ and consider the set $\{z \in [a, b] : f(z) \leq y\}$. Since $f(a) < y$ it is non-empty and since it is a subset of $[a, b]$ it must be bounded. Therefore this set has a finite supremum. So we let $x = \sup\{z \in [a, b] : f(z) \leq y\}$ and note that $x \in [a, b]$. Thus we can find, $(x_n)_{n\in\mathbb{N}}$ a sequence of elements in $[a, b]$ where $f(x_n) \leq y$ for each $n \in \mathbb{N}$ and where $\lim_{n\to\infty} x_n = x$. Since f is continuous we must have that $\lim_{n\to\infty} f(x_n) = f(x)$ and so $f(x) \leq y$.

Since $f(b) > y$ we know that $x \neq b$ and so we consider the non-empty open interval (x, b) and note that for all $z \in (x, b)$ we will have that $f(z) > y$. We can find a sequence $(z_n)_{n\in\mathbb{N}}$ where each $z_n \in (x, b)$ and $\lim_{n\to\infty} z_n = x$. Thus since f is continuous we have that $\lim_{n\to\infty} f(z_n) = f(x)$. Since each element $z_n > x$ we must have that $f(z_n) > y$ for all $n \in \mathbb{N}$ and so $f(x) \geq y$. Therefore $f(x) = y$ and since $y \in (f(a), f(b))$, $x \notin \{a, b\}$ and so $x \in (a, b)$. □

It is easy to see that an analogous statement holds when $f(a) > f(b)$.

Corollary 7.19. *Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ be a continuous function where $f(a) > f(b)$. We have that for all $y \in (f(b), f(a))$ there exists $x \in (a, b)$ such that $f(x) = y$.*

Proof. Consider the function $-f : [a,b] \to \mathbb{R}$. This is continuous and $-f(a) < -f(b)$. Thus if $y \in (f(b), f(a))$ then $-y \in (-f(a), -f(b))$ and so by the Intermediate value theorem there exists $x \in (a,b)$ such that $-f(x) = -y$ and thus $f(x) = y$. $\square$

Note that when we refer to the intermediate value theorem we will often mean the combined statement of Theorem 7.18 and Corollary 7.19. We now turn to some consequences of the intermediate value theorem.

Example. *We will claim that the polynomial $x^3 + 4x - 1$ has a root in $(0,1)$. To do this we will use the intermediate value theorem. We know that if we let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = x^2 + 4x - 1$ then this function is continuous on $[0,1]$. Since $f(0) = -1$ and $f(1) = 4$ we know by the intermediate value there exists $x \in (0,1)$ with $f(x) = 0$. This value x is a root of the polynomial $x^3 + 4x + 1$.*

Corollary 7.20. *If $f, g : [0,1] \to \mathbb{R}$ are continuous functions and $f(0) > g(0)$ but $f(1) < g(1)$ then there exists $x \in (0,1)$ such that $f(x) = g(x)$.*

Proof. Consider the function $f - g : [0,1] \to \mathbb{R}$. This is continuous on $[0,1]$, $(f-g)(0) > 0$ and $(f-g)(1) < 0$. Thus by the Intermediate value theorem there exists $x \in (0,1)$ such that $(f-g)(x) = 0$ and thus $f(x) = g(x)$. $\square$