The Art and Beauty of Pure Mathematics

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THE ART AND BEAUTY OF PURE MATHEMATICS

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Pure mathematics is about understanding structure: seeing order in what initially appears to be chaos.

The Sierra Nevada Mountains, California
(where my dad would often drive us – only 8 hours each way!)

I spent many hours in that car looking at California orchards, and I would wonder, why are the trees arranged so randomly?

And then, for an instant, the car would be aligned just right so that order became clear.

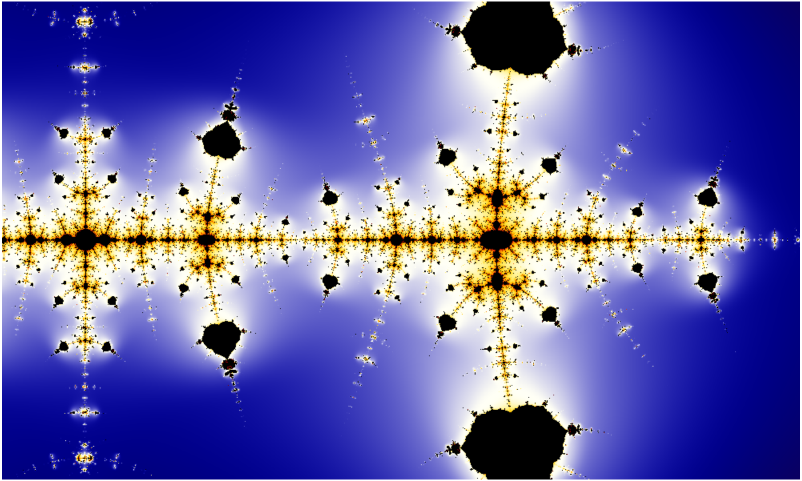
By Sarah Stierch (Flickr: Olive Trees at B.R. Cohn Winery & Vineyard)

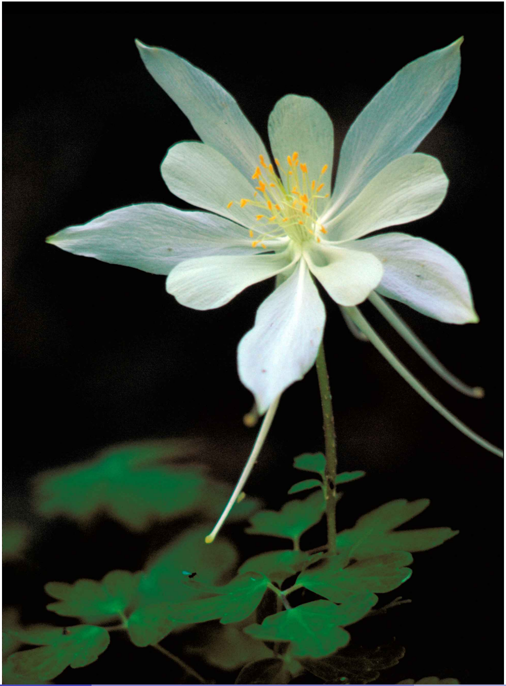
Stepping back can help us see the order and patterns within a system.

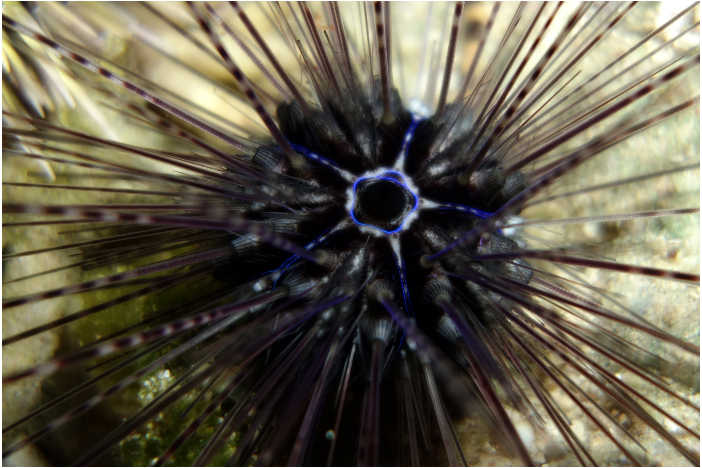
**These insights, these moments of clarity,
are what get mathematicians addicted to math.**

What do we perceive as order? What do we perceive as beauty?

Symmetry is commonly perceived as ordered and beautiful.







By Patrick Randall (Diadema savignyi (Audouin, 1829))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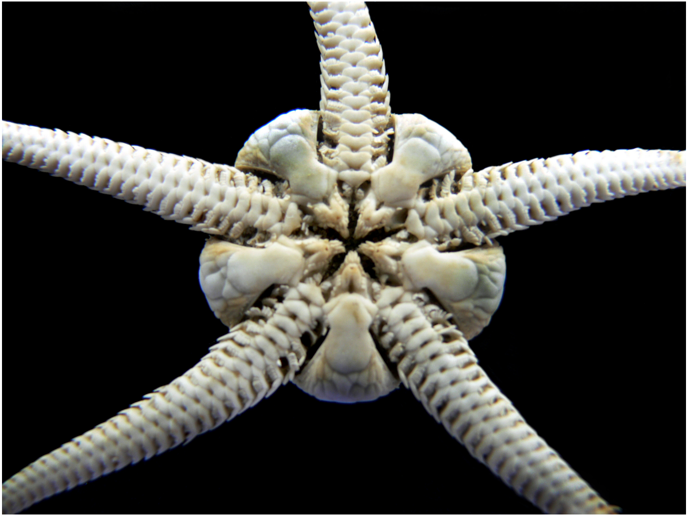

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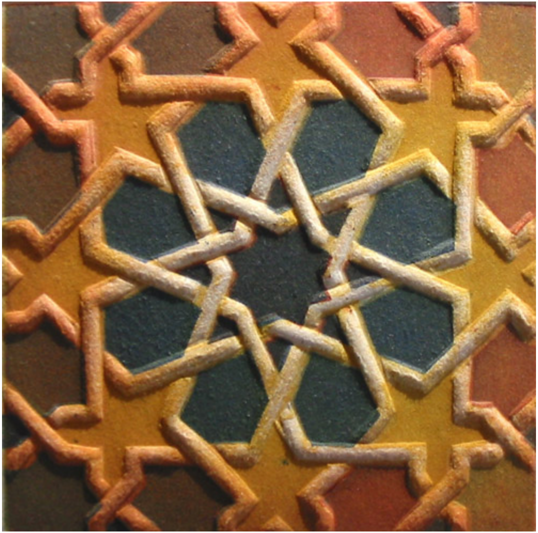

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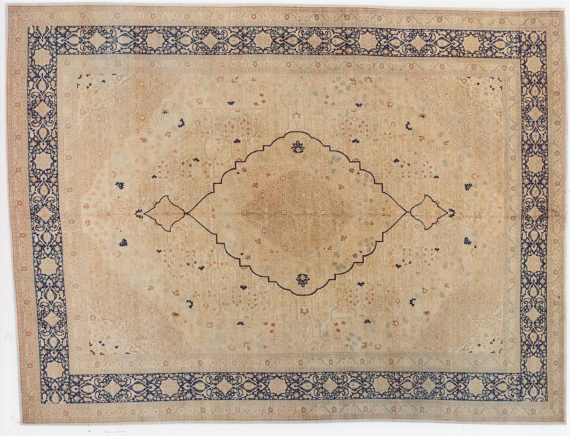
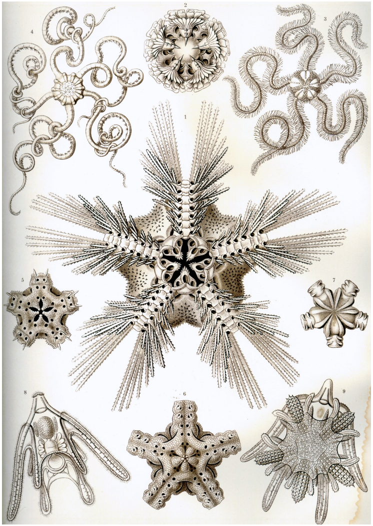


J&D Oriental Rugs Co.

Okay, symmetry can be beautiful, but...

Maybe it's more interesting if we introduce a bit of dissonance?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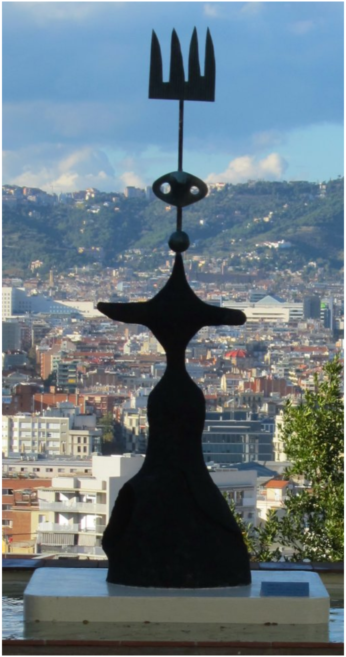


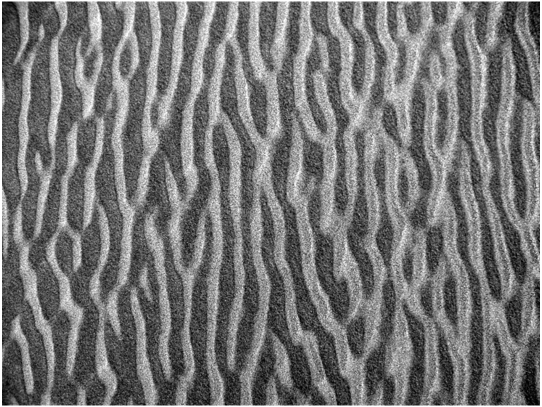
Too much dissonance?

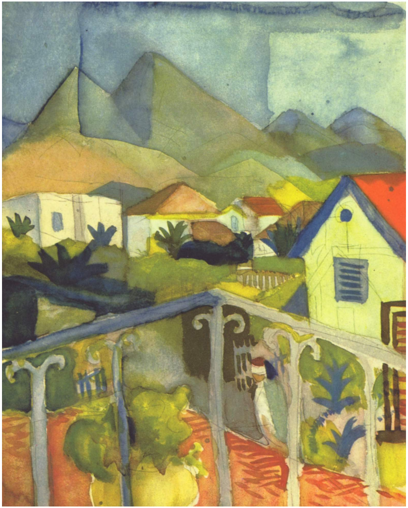
Definitely too much dissonance!

By alf.melin (Flickr)

Hypothesis:

The balance between symmetry and dissonance
affects our perception of beauty.





Hypothesis:

Structure and elegance contribute to our perception of beauty.

So these are my aesthetics when judging the art and beauty in math formulas and proofs:

The balance of symmetry and dissonance, as well as structure and elegance.

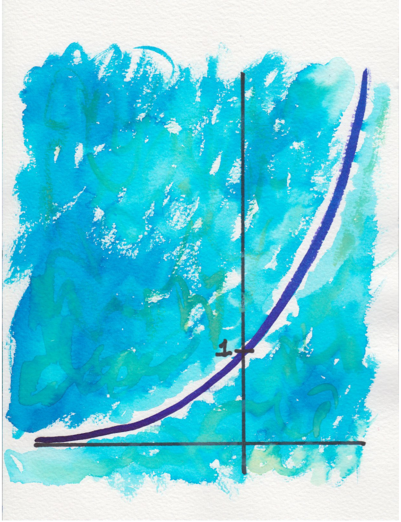
As a random comment: Sometimes randomness is beautiful.

By Badics (Own work),

What are some well-known beautiful theorems in mathematics?

There are some beautiful mathematical identities involving the fundamental functions sin, cos, and exp :

$$y = \cos(x)$$
$$y = \sin(x)$$



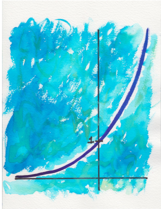
$y = \exp(x)$

Defining $2! = 2 \cdot 1,\ 3! = 3 \cdot 2 \cdot 1,\ 4! = 4 \cdot 3 \cdot 2 \cdot 1,$ et cetera, we have the *Taylor series*:

$$\boxed{\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots}$$

$$\boxed{\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots}$$

$$\boxed{\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \frac{x^9}{9!} + \cdots}$$



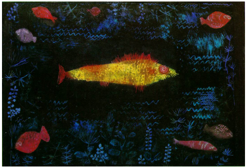
$$\boxed{\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \frac{x^9}{9!} + \cdots}$$

Thus with $i = \sqrt{-1}$ (so $i^2 = -1,\ i^3 = -i,\ i^4 = 1,\ i^5 = i$ etc.), we have

$$\exp(ix) = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \frac{x^8}{8!} + i\frac{x^9}{9!} - \frac{x^{10}}{10!} - i\frac{x^{11}}{11!} + \cdots$$

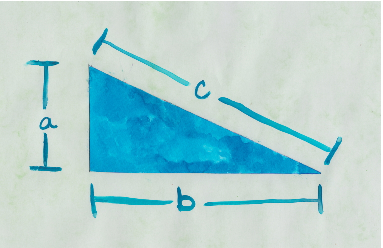
Hence

$$\boxed{\exp(ix) = \cos(x) + i\sin(x).}$$



Many people have cited the Pythagorean Theorem as one of the most beautiful theorems. It states that, given a right triangle with legs of lengths a and b, and hypotenuse of length c, we have

$$\boxed{a^2 + b^2 = c^2.}$$

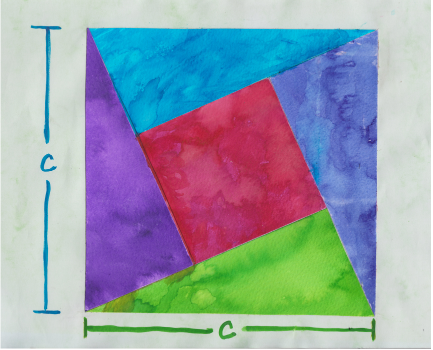


(Notice here I have assumed that $a \leq b$.)

There are many proofs of this theorem, but here I present one that many find particularly elegant:

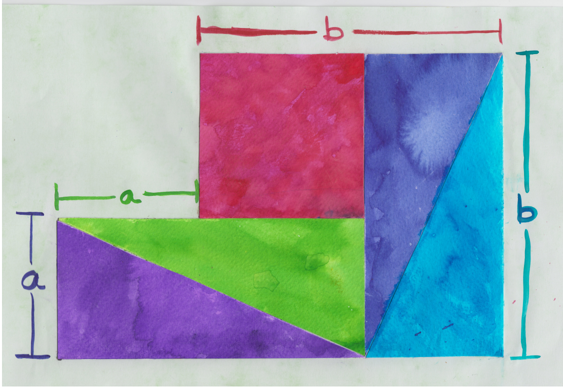
We begin with a square with sides of length c, thus its area is c^2.

We inscribe in this square 4 copies of the right triangle with hypotenuse of length c and legs of lengths a and b:



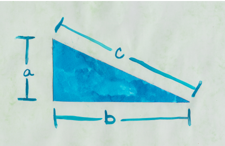
Having assumed that $a \leq b$, this partitioning of the original square includes an inner square with sides of length $b - a$.

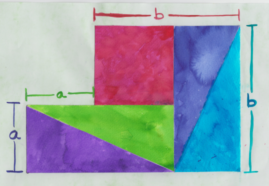
We rotate 2 of the inscribed triangles to obtain the following:



We can view this shape as 2 squares, one of side length a, and the other of side length b. So the total area of this geometric shape is

$$\boxed{a^2 + b^2.}$$





This last geometric shape was made from our original square with area c^2, so we must have

$$\boxed{a^2 + b^2 = c^2,}$$

thus proving the theorem. $\square$

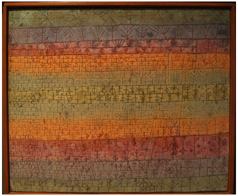
Positive Integers: $1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11, \ldots$

A triple (a, b, c) of *positive integers* with $a^2 + b^2 = c^2$ is called a *Pythagorean triple.*

Example: Take $a = 3$, $b = 4$, and $c = 5$. Then

$$a^2 + b^2 = 3^2 + 4^2 = 25 = 5^2 = c^2.$$

Question: Are there infinitely many Pythagorean triples?



Pythagorean triples stratify into *families*, each of which is generated by a Pythagorean triple (a, b, c) that is *primitive*, meaning that the highest common factor of a and b is 1 (such as the triple $(3, 4, 5)$).

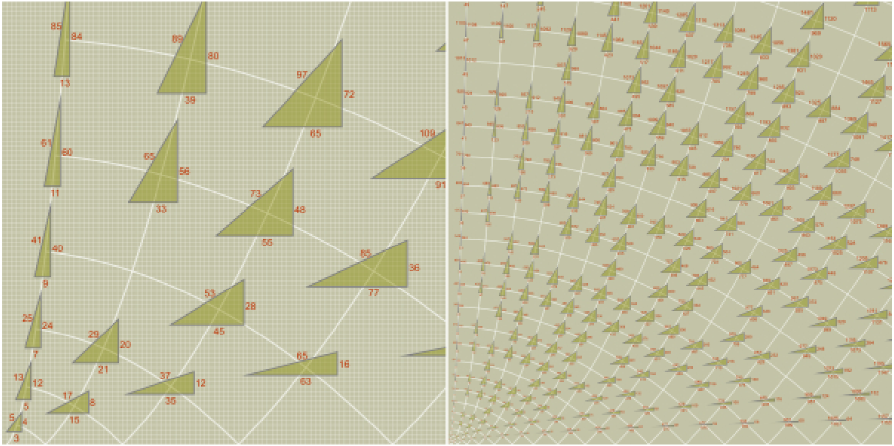
With (a, b, c) a primitive Pythagorean triple, its family consists of all triples of the form

$$(ak, bk, ck)$$

where k varies over all positive integers (being $1,\ 2,\ 3, \dots$).

Euclid developed an algorithm for generating *infinitely many* primitive Pythagorean triples.

Graphs of Primitive Pythagorean Triples



As mathematicians are wont to do, let's modify our preceding question:

How many ways can we write a positive integer n
as a sum of 2 squares of integers?

The Integers: $0,\ 1,\ -1,\ 2,\ -2,\ 3,\ -3,\ 4,\ -4,\ 5,\ -5,\ 6,\ -6, \ldots$

More generally: How many ways can we write a positive integer n
as a sum of k squares of integers?

For instance, with $n = 12$ and $k = 8$, we have:

$$\begin{aligned} 12 &= 3^2 + 1^2 + 1^2 + 1^2 + 0^2 + 0^2 + 0^2 + 0^2 \\ &= 2^2 + 2^2 + 2^2 + 0^2 + 0^2 + 0^2 + 0^2 + 0^2 \\ &= 2^2 + 2^2 + 1^2 + 1^2 + 1^2 + 1^2 + 0^2 + 0^2. \end{aligned}$$

The Integers: $0,\ 1,\ -1,\ 2,\ -2,\ 3,\ -3,\ 4,\ -4,\ 5,\ -5,\ 6,\ -6, \ldots$

The Integers: 0, 1, -1, 2, -2, 3, -3, 4, -4, 5, -5, 6, $-6, \ldots$

$$\boxed{12 = 3^2 + 1^2 + 1^2 + 1^2 + 0^2 + 0^2 + 0^2 + 0^2}$$

When we consider changes of signs and rearrangements of the order of the squares, the above solution gives us other solutions, such as:

$$\begin{aligned} 12 &= (-3)^2 + 1^2 + 1^2 + 1^2 + 0^2 + 0^2 + 0^2 + 0^2 \\ &= 1^2 + 3^2 + 1^2 + 1^2 + 0^2 + 0^2 + 0^2 + 0^2. \end{aligned}$$

Altogether, we find that there are **31,808** (ordered) **8**-tuples of integers so that the **sum of their squares** gives us **12**.

Notation: For n a positive integer, we use $\mathbf{r_8(n)}$ to denote the number of ways we can realise n as the (ordered) sum of 8 squares of integers.

Formula:

$$\boxed{r_8(n) = 16 \sum_{d|n} (-1)^{n-d}\, d^3}$$

where the notation

$$\sum_{d|n}$$

means the sum over all positive integers d that divide n.

$$\boxed{r_8(n) = 16 \sum_{d|n} (-1)^{n-d}\, d^3}$$

With $n = 12$, the positive divisors d of 12 are $1,\ 2,\ 3,\ 4,\ 6,\ 12,$ so

$$\sum_{d|12} (-1)^{12-d}\, d^3 = -1 + 8 - 27 + 64 + 216 + 1728 = 1988$$

thus $$\mathbf{r_8(12)} = 16 \cdot 1988 = \mathbf{31,808}.$$

There is a similar, but slightly more complicated, formula for the number of ways we can realise n as the (ordered) sum of 6 squares of integers:

$$\boxed{r_6(n) = 16 \sum_{d|n} \chi(n/d)\, d^2 - 4 \sum_{d|n} \chi(d)\, d^2}$$

where

$$\chi(d) = \begin{cases} 1 & \text{if } 4 \text{ divides } d-1, \\ -1 & \text{if } 4 \text{ divides } d-3, \\ 0 & \text{if } d \text{ is even.} \end{cases}$$

$$\boxed{r_6(12) = 16\sum_{d|12}\chi(12/d)\,d^2 - 4\sum_{d|12}\chi(d)\,d^2}$$

$$\boxed{\chi(d) = \begin{cases} 1 & \text{if } 4 \text{ divides } d-1, \\ -1 & \text{if } 4 \text{ divides } d-3, \\ 0 & \text{if } d \text{ is even.} \end{cases}}$$

$\chi(1) = 1,\ \chi(2) = 0,\ \chi(3) = -1,\ \chi(4) = 0,\ \chi(6) = 0,\ \chi(12) = 0.$

$$\mathbf{r_6(12)} = 16(-4^2 + 12^2) - 4(1^2 - 3^2) = \mathbf{2080}.$$

More formulas:

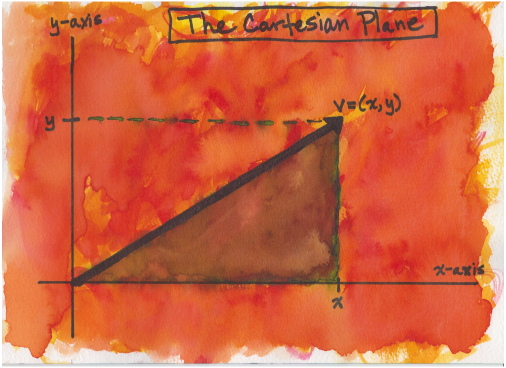
$$\mathbf{r_2(n)} = 4\sum_{d|n}\chi(d), \ \ \mathbf{r_4(n)} = 8(2+(-1)^n)\sum_{d|n,\, d \text{ odd}} d,$$

$$\mathbf{r_{10}(n)} = \frac{64}{5}\sum_{d|n}\chi(n/d)d^4 + \frac{4}{5}\sum_{d|n}\chi(d)d^4 + \textbf{noise}$$

$$\mathbf{r_{12}(n)} = 8(-1)^{n-1}\sum_{d|n}(-1)^{d+n/d}d^5 + \textbf{noise}$$

and the noise gets (relatively) very small as n gets large.

Why is it natural to consider sums of squares?



Capturing geometry: Given a point $v = (x, y)$ in 2 dimensions, we write $\vec{v}$ to denote the vector connecting the origin $(0, 0)$ and (x, y). By the *PythagoreanTheorem*, the (square of the) length of the vector $\vec{v}$ is $x^2 + y^2$.

More generally, in m dimensions, we identify each point v with an ordered m-tuple of numbers

$$(x_1, x_2, x_3, \ldots, x_m),$$

relative to (perpendicular) coordinate axes. Then we write $\vec{v}$ to refer to the *vector* from the origin $(0, 0, 0, \ldots, 0)$ to $v = (x_1, x_2, x_3, \ldots, x_m)$, and we define the dot product of $\vec{v}$ with itself by

$$\vec{v} \cdot \vec{v} = x_1^2 + x_2^2 + x_3^2 + \cdots + x_m^2.$$

By the *generalised* Pythagorean Theorem, this is the square of the length of $\vec{v}$.

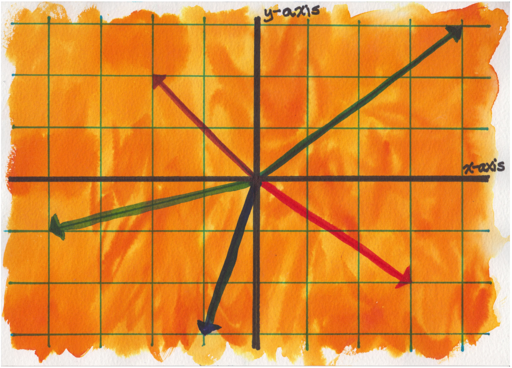
Further, with $v = (x_1, x_2, x_3, \ldots, x_m)$ and $w = (y_1, y_2, y_3, \ldots, y_m)$, we extend the definition of the dot product to give us

$$\vec{v} \cdot \vec{w} = x_1 y_1 + x_2 y_2 + x_3 y_3 + \cdots + x_m y_m,$$

and we find that $\vec{v}$ and $\vec{w}$ are **perpendicular** exactly when $\vec{v} \cdot \vec{w} = 0$.

The integers: 0, 1, −1, 2, −2, 3, −3, 4, −4, 5, −5, 6, −6, ...

Why is it natural to only consider sums of squares of integers?



Many systems are *discrete*, as in the digital world, so these systems can be modeled using *lattices*, in which the vectors join the origin to points $(x_1, x_2, \ldots, x_m)$ where the coordinates $x_1, x_2, x_3 \ldots, x_m$ are integers.

From whence come these formulas for $\mathbf{r_k(n)}$?

For sums of 6, 8, 10, or 12 squares, these formulas come from the *theory of modular forms and theta series.*

With z a (complex) variable, k a positive integer, we set

$$\theta_k(z) = \sum_{n\geq 0} \mathbf{r_k(n)}\, z^n.$$

Using the geometry captured by the dot product, we can prove that $\theta_k(z)$ is a particular sort of function called a *modular form of "weight" $k/2$*.

How does this help?

When $k = 8$, because $\theta_8(z)$ is a *modular form of weight* 4, we have

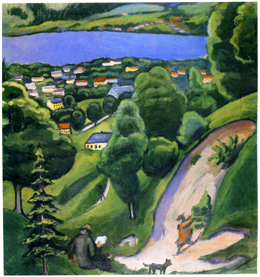
$$\theta_8(z) = \sum_{n \geq 0} \mathbf{r_8(n)}\, z^n = aE_1(z) + bE_2(z) + cE_3(z)$$

for some numbers a, b, c, where

$$E_1(z) = (constant) + \sum_{n>0} \left(\sum_{d|n,\, d \text{ odd}} d^3 \right) z^n,$$

$$E_2(z) = (constant) + \sum_{n>0} \left(\sum_{d|n,\, n/d \text{ odd}} d^3 \right) z^n,$$

$$E_3(z) = (constant) + \sum_{n>0} \left(\sum_{d|n,\, d, n/d \text{ odd}} d^3 \right) z^n.$$



By **brute force**, we can compute that

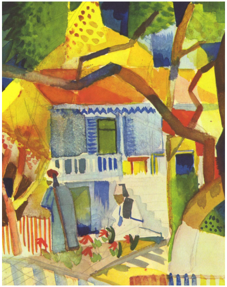
$$r_8(1) = 16, \; r_8(2) = 16 \cdot 7, \; r_8(4) = 16 \cdot 71.$$

So, matching coefficients, the only way we can have

$$\theta_8(z) = \sum_{n \geq 0} \mathbf{r_8(n)}\, z^n = aE_1(z) + bE_2(z) + cE_3(z)$$

is with

$$a = -\frac{240}{7}, \; b = \frac{128}{7}, \; c = 32.$$



From this we derive the formula

$$\boxed{r_8(n) = 16 \sum_{d|n} (-1)^{n-d}\, d^3}$$

The process for finding the formula for $r_6(n)$ is similar, but finding the formulas for $r_{10}(n)$ and $r_{12}(n)$ is a bit more *sneaky*...

For sums of 10 squares: We perform a rather subtle *averaging* to get

$$\widetilde{\theta}_{10}(z) = \theta_{10}(z) + \text{noise} = \sum_{n\geq 0} \mathbf{r_{10}(n)} z^n + \text{noise}$$

and because we can show that $\widetilde{\theta}_{10}(z)$ is a particularly **nice** modular form (of weight 5), we have that

$$\widetilde{\theta}_{10}(z) = aE_1(z) + bE_2(z)$$

for some numbers a and b where

$$E_1(z) = (constant) + \sum_{n>0} \left(\sum_{d|n} \chi(n/d)\, d^4 \right) z^n,$$

$$E_2(z) = (constant) + \sum_{n>0} \left(\sum_{d|n} \chi(d)\, d^4 \right) z^n.$$

Then we use information about the behaviour of $\widetilde{\theta}_{10}(z)$, $E_1(z)$, $E_2(z)$ *at infinity* to find the values of a and b, giving us

$$\widetilde{\theta}_{10}(z) = \frac{64}{5} E_1(z) + \frac{4}{5} E_2(z)$$

and hence

$$\boxed{\theta_{10}(z) = \frac{64}{5} E_1(z) + \frac{4}{5} E_2(z) + \text{noise}.}$$

My Research:

By changing how we define the dot product we change the **geometry** on our mother lattice.

I consider **all** geometries (in which nonzero vectors have positive lengths). Then I ask, how many vectors in the mother lattice have a given length?

More generally I ask, how many *sublattices* Λ of the mother lattice have a given geometry?

Sometimes I can't get formulas for these counts, but I can get relations among these counts, such as:

$$\boxed{(p^{k-1}+1)(p^{k-2}+1)\cdots(p^{k-t}+1)\,\widetilde{r}(\Lambda) = \sum_{p\Lambda\subset\Omega\subseteq\frac{1}{p}\Lambda} p^{*}\,\widetilde{r}(\Omega).}$$

Here p is a prime (such as $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, \ldots$), $2k$ is the dimension of the mother lattice,
Λ is a lattice in t dimensions with a given geometry, $r(\Lambda)$ is the number of sublattices contained in the mother lattice with the geometry of Λ, and $\widetilde{r}(\Lambda)$ is an average count so that $r(\Lambda) = \widetilde{r}(\Lambda) + \textit{noise}$.

Sometimes being a mathematical researcher feels like being an archaeologist, uncovering structure and patterns that are hidden below the surface. It's exciting and rewarding, and to me, these structures look like art.

Thank you!