Analysis Workshop
Fermat's method to evaluate $\int_0^1 \frac{1}{\sqrt{x}}\,dx$

Story. Like the influential mathematician Bonaventura Cavalieri (1598-1647), Pierre de Fermat (1601-1665) set out to compute areas under curves. These days we do this using integrals, but integrals were not formally introduced until the mid-17th century (and now we do not consider what was then introduced to be completely rigourous). An ingenious mathematician, Fermat developed a method for finding the areas under certain curves (and his method can easily be made rigourous, as we will see later in our rigourous development of integrals in Analysis 1).

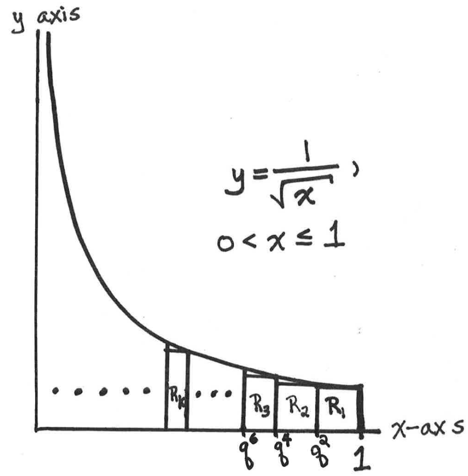
In this workshop you will use Fermat's method to evaluate the integral

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx,$$

which measures the area under the curve $y = \frac{1}{\sqrt{x}}$ for $0 < x \leq 1$. (Note that with $f(x)$ a function defined for all $x \in [0,1]$, the area under the curve $y = f(x)$ for $0 < x \leq 1$ is equal to the area under the curve $y = f(x)$ for $0 \leq x \leq 1$, as these areas differ by the area of a line segment, which is 0.)

At times you will be asked to think like Fermat, and to use your intuition (and experience working with numbers) to make conclusions.

To evaluate the area under the curve $y = 1/\sqrt{x}$ for $0 < x \leq 1$, we use the following. Fix a real number q so that $0 < q < 1$. For $k = 1, 2, 3, \ldots$, let R_k be the rectangle in the Cartesian plane whose base is on the x-axis between $x = q^{2k}$ and q^{2k-2}, and whose height is $1/\sqrt{q^{2k-2}} = q^{-k+1}$. Let a_k be the area of R_k. (Note that since $0 < q < 1$, we have $q^{2k} < q^{2k-2}$.)



We approximate this area by first summing the areas of more and more (and then infinitely many) rectangles, and then taking the width of the rectangles to get smaller and smaller.

For further exploration: What other areas under curves can be evaluated using Fermat's method? For instance, by taking $q > 1$, could one use this method to find the area under the curve $y = 1/x^2$ for $1 \leq x < \infty$? Could one modify this method to find the area under the curve $y = 1/x^3$ for $1 \leq x < \infty$?