INTRODUCTION TO PROOFS: HW1 SOLUTIONS

Your solutions should be organised to proceed logically, and should be written in complete sentences.

Note: In the solutions, remarks made in square brackets [such as these] are not necessary for a complete proof.

1.2. Let $X = \{x \in \mathbb{R} : \quad x \neq 1\}$, $Y = \{y \in \mathbb{R} : \quad y \neq 3\}$. Define $f : X \to Y$ by $f(x) = \frac{3x}{x-1}$. **Fact:** f does in fact map X into Y, and f is surjective.

(a) Find a function $g : Y \to X$ so that $g \circ f$ is the identity function on X. (In your scratch work, to find g you may want to set $y = f(x)$ and solve for x in terms of y; however, in your solution, you should begin by defining g and then proceed to prove that for every $y \in Y$, we have $g(y) \in X$, and for every $x \in X$, we have $g \circ f(x) = x$.)

(b) Show that g is surjective. (In your scratch work, you may want to begin by setting $x = g(y)$ and then solving for y. However, in your presentation, you should begin by choosing (arbitrary) $x \in X$, then simply produce the value for y and demonstrate that $y \in Y$ with $g(y) = x$.)

(c) Show that $f \circ g$ is the identity map on Y.

Solution: (a) [Scratch work: Set $y = \frac{3x}{x-1}$. So $yx - y = y(x-1) = 3x$; then $y = yx - 3x = x(y-3)$, and hence $x = \frac{y}{y-3}$.]

Define $g : Y \to X$ by $g(y) = \frac{y}{y-3}$. So for any $y \in Y$, $g(y) \in \mathbb{R}$, and since $y \neq y - 3$, we have $\frac{y}{y-3} \neq 1$. Thus g does indeed map Y into X. Now take $x \in X$; then

$$g \circ f(x) = g(f(x)) = \frac{f(x)}{f(x) - 3} = \frac{\frac{3x}{x-1}}{\frac{3x}{x-1} - 3} = \frac{3x}{3x - 3(x-1)} = x.$$

Thus $g \circ f$ is the identity map on X.

(b) [Scratch work: Suppose $x = g(y)$. Thus $x = \frac{y}{y-3}$, so $x(y-3) = y$; thus $y(x-1) = 3x$ and so $y = \frac{3x}{x-1}$.]

Choose [arbitrary] $x \in X$ [so $x \neq 1$]. Take $y = \frac{3x}{x-1} = 3 \cdot \frac{x}{x-1}$. Since $x \neq 1$, we have $y \in \mathbb{R}$; also, $y \neq 3$ since $x \neq x - 1$ and hence $\frac{x}{x-1} \neq 1$. Thus $y \in Y$. Further,

$$g(y) = \frac{y}{y-3} = \frac{\frac{3x}{x-1}}{\frac{3x}{x-1} - 3} = \frac{3x}{3x - 3(x-1)} = x.$$

Since x was chosen arbitrarily from X, this shows that $g : Y \to X$ is surjective.

ALTERNATIVELY: Take [arbitrary] $x \in X$. Let $y = f(x)$. Then since $g \circ f$ is the identity map on X, we have

$$x = g \circ f(x) = g(f(x)) = g(y).$$

Thus g is surjective.

(c) Take $y \in Y$. Then

$$f \circ g(y) = f(g(y)) = \frac{3g(y)}{g(y) - 1} = \frac{3\frac{y}{y-3}}{\frac{y}{y-3} - 1} = \frac{3y}{y - (y-3)} = y.$$

Thus $f \circ g$ is the identity map on Y.

1.5. (a) Suppose $f : X \to Y$, $g : Y \to Z$. Show that if f and g are surjective then so is $g \circ f$. (Begin by choosing [arbitrary] $z \in Z$. You must show there is some $x \in X$ so that $g \circ f(x) = z$. Use first that g is surjective.)

Solutions:

(a) Choose $z \in Z$. [So the only condition on z is that it lies in Z; that is, z is an arbitrary element of Z.] Since g is surjective, there is some $y \in Y$ so that $g(y) = z$. [Note that y is **not** an arbitrary element of Y, as y must meet the condition $g(y) = z$.] Since f is surjective, there is some $x \in X$ so that $f(x) = y$. Thus we have

$$g \circ f(x) = g(f(x)) = g(y) = z.$$

[Note: The order of unwinding the expression $g \circ f(x)$ is important to produce a correct argument.] Thus we have shown that for any $z \in Z$, there is some $x \in X$ so that $g \circ f(x) = z$. This shows that $g \circ f$ is a surjective function from X onto Z.

1.6. Suppose $f : X \to Y$, $g : Y \to X$ so that $g \circ f$ is the identity map on X (so for all $x \in X$, we have $g \circ f(x) = x$). Suppose f is surjective; prove that $f \circ g$ is the identity map on Y. (Suggestion: Take $x \in X$; evaluate $f \circ g \circ f(x)$ in two ways. Now take $y \in Y$; use that f is surjective and what you have just shown to conclude that $f \circ g(y) = y$.)

Solution: Take $x \in X$. Then

$$f \circ g \circ f(x) = (f \circ g) \circ f(x) = f \circ g(f(x)).$$

Also,

$$f \circ g \circ f(x) = f \circ (g \circ f)(x) = f(g \circ f(x)),$$

and since $g \circ f$ is the identity map on X, $f(g \circ f(x)) = f(x)$. Thus

$$f \circ g(f(x)) = f \circ g \circ f(x) = f(x).$$

Now take $y \in Y$. Since f is surjective, there is some $x \in X$ so that $f(x) = y$; hence

$$f \circ g(y) = f \circ g(f(x)) = f(x) = y.$$

As this holds $\forall y \in Y$, this shows that $f \circ g$ is the identity map on Y.