INTRODUCTION TO PROOFS: HW2 SOLUTIONS

Your solutions should be organised to proceed logically, and should be written in complete sentences.

Note: In the solutions, remarks made in square brackets [such as these] are not necessary for a complete proof.

1.7. Suppose $f : X \to Y$ is bijective.

(a) Suppose $g : Y \to Z$ is bijective (and hence we know $g \circ f$ is bijective). Show that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ (here $(g \circ f)^{-1}$ denotes the inverse of $g \circ f$, which we have seen is unique). (So you need to show that for any $z \in Z$, we have $(g \circ f)^{-1}(z) = f^{-1} \circ g^{-1}(z)$. Take $y \in Y$ so that $g^{-1}(z) = y$, and take $x \in X$ so that $f^{-1}(y) = x$. Recall that we have a "recipe" for describing an inverse function.)

Solution:

(a) We know $(g \circ f)^{-1} : Z \to X$ and $f^{-1} \circ g^{-1} : Z \to X$. Choose [arbitrary] $z \in Z$. Take $y \in Y$ so that $g^{-1}(z) = y$, and take $x \in X$ so that $f^{-1}(y) = x$. Thus $z = g(y)$ and $y = f(x)$. Hence we also have

$$g \circ f(x) = g(f(x)) = g(y) = z,$$

so $(g \circ f)^{-1}(z) = x$. Also,

$$f^{-1} \circ g^{-1}(z) = f^{-1}(g^{-1}(z)) = f^{-1}(y) = x.$$

[NOTE: The order of unwinding $f^{-1} \circ g^{-1}(z)$ is very important to produce a correct argument.] Hence we have shown that for any $z \in Z$, we have $(g \circ f)^{-1}(z) = f^{-1} \circ g^{-1}(z)$, so $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

2.1. Suppose P, Q, R are propositions.

(c) Show that $P \vee (Q \wedge R) \iff (P \vee Q) \wedge (P \vee R)$.

Solution:

(c) We have

P	Q	R	$(Q \wedge R)$	$[P \vee (Q \wedge R)]$
T	T	T	T	T
T	T	F	F	T
T	F	T	F	T
T	F	F	F	T
F	T	T	T	T
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

Also:

P	Q	R	$(P \vee Q)$	$(P \vee R)$	$[(P \vee Q) \wedge (P \vee R)]$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	F	T	F
F	F	F	F	F	F

So for any combination of truth values for P, Q, R, the truth values of $[P \vee (Q \wedge R)]$ and $[(P \vee Q) \wedge (P \vee R)]$ agree. Thus $[P \vee (Q \wedge R)] \iff [(P \vee Q) \wedge (P \vee R)]$.

2.2 Suppose P, Q are propositions. Show:
(b) $\neg(P \implies Q) \iff (P \wedge \neg Q)$.

Solution: (a)

P	Q	$(P \implies Q)$	$(\neg P \implies Q)$	$\neg Q$	$(P \wedge \neg Q)$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F

So for any combination of truth values for P, Q, the truth values of $\neg(P \implies Q)$ and $(P \wedge \neg Q)]$ agree. Thus $\neg(P \implies Q) \iff (P \wedge \neg Q)]$.

2.5. Suppose P, Q, R are propositions.
(a) Show that $(P \implies Q) \iff R$ is not equivalent to $P \implies (Q \iff R)$.

Solution: (a)

P	Q	R	$[(P \implies Q) \iff R]$	$[P \implies (Q \iff R)]$
T	T	T	T	T
T	T	F	F	F
T	F	T	F	F
T	F	F	T	T
F	T	T	T	T
F	T	F	F	T
F	F	T	T	T
F	F	F	F	T

Thus when P and R are both false, we see that $(P \implies Q) \iff R$ and $P \implies (Q \iff R)$ do not have the same truth values. Hence $(P \implies Q) \iff R$ and $P \implies (Q \iff R)$ are not equivalent.

3.2. Negate the following propositions:

(a) $\exists i \in I$ so that $x \in B_i$,
(b) $\exists c \in \mathbb{R}$ so that $\forall \varepsilon > 0,\ \exists N \in \mathbb{Z}_+$ so that $\forall n \geq N,\ |a_n - c| < \varepsilon$.
(c) $\forall \varepsilon > 0,\ \exists N \in \mathbb{Z}_+$ so that $\forall n \geq N,\ \forall m \geq N,\ |a_n - a_m| \leq \varepsilon$.
(d) $\forall f : X \to Y,\ \forall A \subseteq X,\ \forall B \subseteq X,\ f(A \smallsetminus B) \not\subseteq f(A) \smallsetminus f(B)$.

Solution:

(a) $\forall i \in I,\ x \notin B_i$.

(b)

$$\begin{aligned}
&\neg[\exists c \in \mathbb{R} \text{ so that } \forall \varepsilon > 0,\ \exists N \in \mathbb{Z}_+ \text{ so that } \forall n \geq N,\ |a_n - c| < \varepsilon] \\
&\iff \forall c \in \mathbb{R},\ \neg[\forall \varepsilon > 0,\ \exists N \in \mathbb{Z}_+ \text{ so that } \forall n \geq N,\ |a_n - c| < \varepsilon] \\
&\iff \forall c \in \mathbb{R},\ \exists \varepsilon > 0 \text{ so that } \neg[\exists N \in \mathbb{Z}_+ \text{ so that } \forall n \geq N,\ |a_n - c| < \varepsilon] \\
&\iff \forall c \in \mathbb{R},\ \exists \varepsilon > 0 \text{ so that } \forall N \in \mathbb{Z}_+,\ \neg[\forall n \geq N,\ |a_n - c| < \varepsilon] \\
&\iff \forall c \in \mathbb{R},\ \exists \varepsilon > 0 \text{ so that } \forall N \in \mathbb{Z}_+,\ \exists n \geq N \text{ so that } \neg[|a_n - c| < \varepsilon] \\
&\iff \forall c \in \mathbb{R},\ \exists \varepsilon > 0 \text{ so that } \forall N \in \mathbb{Z}_+,\ \exists n \geq N \text{ so that } |a_n - c| \geq \varepsilon.
\end{aligned}$$

(c)

$$\begin{aligned}
&\neg[\forall \varepsilon > 0,\ \exists N \in \mathbb{Z}_+ \text{ so that } \forall n \geq N,\ \forall m \geq N,\ |a_n - a_m| \leq \varepsilon] \\
&\iff \exists \varepsilon > 0 \text{ so that } \neg[\exists N \in \mathbb{Z}_+ \text{ so that } \forall n \geq N,\ \forall m \geq N,\ |a_n - a_m| \leq \varepsilon] \\
&\iff \exists \varepsilon > 0 \text{ so that } \forall N \in \mathbb{Z}_+,\ \neg[\forall n \geq N,\ \forall m \geq N,\ |a_n - a_m| \leq \varepsilon] \\
&\iff \exists \varepsilon > 0 \text{ so that } \forall N \in \mathbb{Z}_+,\ \exists n \geq N \text{ so that } \neg[\forall m \geq N,\ |a_n - a_m| \leq \varepsilon] \\
&\iff \exists \varepsilon > 0 \text{ so that } \forall N \in \mathbb{Z}_+,\ \exists n \geq N \text{ so that } \exists m \geq N \text{ with } \neg[|a_n - a_m| \leq \varepsilon] \\
&\iff \exists \varepsilon > 0 \text{ so that } \forall N \in \mathbb{Z}_+,\ \exists n \geq N \text{ so that } \exists m \geq N \text{ with } |a_n - a_m| > \varepsilon.
\end{aligned}$$

(d)

$$\begin{aligned}
&\neg[\forall f : X \to Y,\ \forall A \subseteq X,\ \forall B \subseteq X,\ f(A \smallsetminus B) \not\subseteq f(A) \smallsetminus f(B)] \\
&\iff \exists f : X \to Y \text{ so that } \neg[\forall A \subseteq X,\ \forall B \subseteq X,\ f(A \smallsetminus B) \not\subseteq f(A) \smallsetminus f(B)] \\
&\iff \exists f : X \to Y,\ \exists A \subseteq X, \text{ so that } \neg[\forall B \subseteq X,\ f(A \smallsetminus B) \not\subseteq f(A) \smallsetminus f(B)] \\
&\iff \exists f : X \to Y,\ \exists A \subseteq X,\ \exists B \subseteq X \text{ so that } \neg[f(A \smallsetminus B) \not\subseteq f(A) \smallsetminus f(B)] \\
&\iff \exists f : X \to Y,\ \exists A \subseteq X,\ \exists B \subseteq X \text{ so that } f(A \smallsetminus B) \subseteq f(A) \smallsetminus f(B).
\end{aligned}$$