INTRODUCTION TO PROOFS: HW3 SOLUTIONS

Your solutions should be organised to proceed logically, and should be written in complete sentences.

Note: In the solutions, remarks made in square brackets [such as these] are not necessary for a complete proof.

3.3. (b) Let $(0,1) = \{x \in \mathbb{R} : \ 0 < x < 1\ \}$. Define $f : (0,1) \to \mathbb{R}$ by $f(x) = \frac{1-x}{x}$. Show that f is injective.

Solutions:

(b) Suppose $x_1, x_2 \in (0,1)$ so that $f(x_1) = f(x_2)$. Thus

$$\frac{1}{x_1} - 1 = \frac{1-x_1}{x_1} = \frac{1-x_2}{x_2} = \frac{1}{x_2} - 1.$$

Hence

$$\frac{1}{x_1} = \frac{1}{x_2},$$

so multiplying this equation by x_1x_2, we get $x_2 = x_1$. Thus, if $\exists x_1, x_2 \in (0,1)$ so that $f(x_1) = f(x_2)$, then $x_1 = x_2$. Hence f is injective.

3.5. Suppose $f : X \to Y$, $g : Y \to X$ so that $g \circ f$ is the identity map on X, meaning that for all $x \in X$, we have $g \circ f(x) = x$. Suppose g is injective; prove that $f \circ g$ is the identity map on Y. (Suggestion: Take $y \in Y$. Evaluate $g \circ f \circ g(y)$ in two ways, using that $(g \circ f) \circ g = g \circ f \circ g = g \circ (f \circ g)$; then use that g is injective. Recall that in the notes for this section, we presented the contrapositive of the definition of g being injective; this will be useful in this proof.) [Note: This is proved in the lecture notes by a different method.]

Solution:

Take $y \in Y$; we evaluate $g \circ f \circ g(y)$ in two ways: We have

$$g \circ f \circ g(y) = (g \circ f) \circ g(y) = g \circ f(g(y)).$$

Since $g(y) \in X$ and $g \circ f$ is the identity map on X, we have

$$g \circ f \circ g(y) = g \circ f(g(y)) = g(y).$$

On the hand,

$$g \circ f \circ g(y) = g \circ (f \circ g)(y) = g(f \circ g(y)).$$

Hence we have

$$g(y) = g \circ f \circ g(y) = g(f \circ g(y)).$$

Since g is injective and $g(y) = g(f \circ g(y))$, we must have $y = f \circ g(y)$. As this argument holds for any $y \in Y$, this shows that $\forall y \in Y$, $y = f \circ g(y)$, or equivalently, $f \circ g$ is the identity map on Y.

4.1. Suppose A, B, C are subsets of a set X.

(b) Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. (Suggestion: Show that for $x \in X$, we have $x \in A \cup (B \cap C) \iff x \in (A \cup B) \cap (A \cup C)$.)

Solution:

(b) Suppose $x \in X$. Let P be the propostion $x \in A$, Q the proposition $x \in B$, and R the proposition $x \in C$. Recall that $P \vee (Q \wedge R) \iff (P \vee Q) \wedge (P \vee R)$. Then:

$$\begin{aligned} x \in A \cup (B \cap C) &\iff x \in A \vee x \in B \cap C \\ &\iff x \in A \vee (x \in B \wedge x \in C) \\ &\iff P \vee (Q \wedge R) \\ &\iff (P \vee Q) \wedge (P \vee R) \\ &\iff (x \in A \vee x \in B) \wedge (x \in a \vee x \in C) \\ &\iff (x \in A \cup B) \wedge (x \in A \cup C) \\ &\iff x \in (A \cup B) \cap (A \cup C). \end{aligned}$$

Thus the elements of $A \cup (B \cap C)$ are exactly the elements of $(A \cup B) \cap (A \cup C)$, and hence $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Solution:

(a) [Note: This is almost exactly like the proof in the notes that $A \cap (B \cap C) = (A \cap B) \cap C$.] Suppose $x \in X$. Let P be the proposition $x \in A$, Q the proposition $x \in B$ and R the proposition $x \in C$. Recall that $P \vee (Q \vee R) \equiv (P \vee Q) \vee R$.

$$\begin{aligned} x \in A \cup (B \cup C) &\iff (x \in A) \vee (x \in B \cup C) \\ &\iff (x \in A) \vee (x \in B \vee x \in C) \\ &\iff P \vee (Q \vee R) \\ &\iff (P \vee Q) \vee R \\ &\iff (x \in A \vee x \in B) \vee x \in C \\ &\iff (x \in A \cup B) \vee (x \in C) \\ &\iff x \in (A \cup B) \cup C. \end{aligned}$$

Thus the elements of X that are in $A \cup (B \cup C)$ are exactly the elements of X that are in $(A \cup B) \cup C$, so $A \cup (B \cup C) = (A \cup B) \cup C$.

4.3. Suppose A, B are subsets of a set X.

(a) Prove that $(A \setminus B)^c = A^c \cup B$. (So you must show that $x \notin A \setminus B$ if and only if $x \notin A$ or $x \in B$).

Solution:

(a) Suppose $x \in X$. Thus:

$$\begin{aligned} x \in (A \setminus B)^c &\iff \neg(x \in A\setminus) \\ &\iff \neg(x \in A \wedge x \notin B) \\ &\iff \neg(x \in A) \vee \neg(x \notin B) \\ &\iff (x \notin A) \vee (x \in B) \\ &\iff (x \in A^c) \vee (x \in B). \end{aligned}$$

Thus the elements of X that are in $(A \setminus B)^c$ are exactly the elements of X that are in $A^c \cup B$; hence $(A \setminus B)^c = A^c \cup B$.

4.7. (a) Suppose $f : X \to Y$, and $U \subseteq X$, $V_1, V_2 \subseteq Y$. Show that $f^{-1}(V_1 \cup V_2) = f^{-1}(V_1) \cup f^{-1}(V_2)$.

Solution:

(a) We have

$$\begin{aligned} x \in f^{-1}(V_1 \cup V_2) &\iff f(x) \in V_1 \cup V_2 \\ &\iff f(x) \in V_1 \ \vee \ f(x) \in V_2 \\ &\iff x \in f^{-1}(V_1) \ \vee \ x \in f^{-1}(V_2) \\ &\iff x \in f^{-1}(V_1) \cup f^{-1}(V_2). \end{aligned}$$

Thus $f^{-1}(V_1 \cup V_2) = f^{-1}(V_1) \cup f^{-1}(V_2)$.