# INTRODUCTION TO PROOFS: HOMEWORK ASSIGNMENTS

**Your solutions should be organised to proceed logically, and should be written in complete sentences.**

**Homework Assignment 5 (from the posted Exercises for the course):** Exercises 7.1(a), 7.3(a), 7.4(a), 8.4(a), 8.5(a), 9.3(a),(b).

7.1. Let $a_1, a_2, a_3, \ldots$ be the Fibonacci sequence; so $a_1 = a_2 = 1$, and for $i \in \mathbb{Z}$ with $i \geq 3$, $a_i = a_{i-1} + a_{i-2}$.

(a) Use strong induction to prove that for $n \in \mathbb{Z}_+$,

$$a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

(Suggestion: Show directly that $P(1), P(2)$ hold. Then suppose that $k \in \mathbb{Z}$ with $k \geq 2$, and that $P(i)$ holds for all $i \in \mathbb{Z}_+$ with $i \leq k$, and use this to evaluate $a_k + a_{k-1}$.)

7.3. (a) Find all primes $p$ so that $7p+4$ is the square of a positive integer. (Suggestion: First suppose that $p$ is a prime so that $7p+4 = n^2$ for some $n \in \mathbb{Z}_+$, and deduce constaints on $p$. Then consider all primes $p$ subject to these constraints and determine for which of these $p$ we have that $7p+4 = n^2$ for some $n \in \mathbb{Z}_+$.)

7.4. Suppose $k \in \mathbb{Z}$ with $k \geq 2$, and $m_1, \ldots, m_{k+1} \in \mathbb{Z}_+$ are pairwise relatively prime, meaning that $\text{hcf}(m_i, m_j) = 1$for $i, j \in \mathbb{Z}$ with $1 \leq i \leq k+1$, $1 \leq j \leq k+1$ and $i \neq j$. Set $M = m_1 m_2 \cdots m_k$.

(a) Suppose $p$ is a prime so that $p|M$. Show that $p \nmid m_{k+1}$.

8.4. Suppose $X, Y$ are countable sets.

(a) Show that $X \times Y$ is countable. (Suggestion: either construct a bijective map from $\mathbb{Z}_+ \times \mathbb{Z}_+$ to $X \times Y$, and use that $\mathbb{Z}_+ \times \mathbb{Z}_+$ is countable, or alternatively, using that $\mathbb{Z}_+ \times \mathbb{Z}_+$ is countable and $X \times Y$ is infinite [as shown in the notes], construct an injective map from $X \times Y$ into $\mathbb{Z}_+$. In the first instance, you must **prove** the map is bijective, and then explain why this means $X \times Y$ is countable; in the second instance, you must **prove** the map is injective, and then explain why this means $X \times Y$ is countable.)

8.5. Note that $\mathbb{Z}_+ \subseteq \mathbb{Q}_+ \subseteq \mathbb{Q}$; since $\mathbb{Z}_+$ is infinite, so are $\mathbb{Q}_+, \mathbb{Q}$.

(a) Show that $\mathbb{Q}_+$ is countable. (Suggestion: Recall that

$$\mathbb{Q}_+ = \left\{ \frac{a}{b} : \ a, b \in \mathbb{Z}_+, \ \text{hcf}(a,b) = 1 \right\}.$$

Begin by defining an injective map from $\mathbb{Q}_+$ into $\mathbb{Z}_+ \times \mathbb{Z}_+$. Note that you must **prove** this map is injective, then you must explain why that means $\mathbb{Q}_+$ is countable.)

9.3. Let $A, B$ be sets. Show:
(a) $(A \subseteq B) \iff (\mathcal{P}(A) \subseteq \mathcal{P}(B))$.
(b) $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.