Mathematical Investigations Workshop
Korteweg-de Vries equation

Story. In this workshop we will investigate certain solutions of the Korteweg-de Vries (KdV) equation, a differential equation that among other things describes waves in shallow water. The KdV equation was motivated by experiments by John Scott Russell in 1834. It was first introduced in a footnote by Joseph Valentin Boussinesq in 1877 and then rediscovered by Diederik Korteweg and Gustav de Vries in 1895. (Like many things it is not named after the person who discovered it first.) However the KdV equation was not studied a lot until in 1965 computer experiments by Zabusky and Kruskal showed solutions that behaved in a peculiar way. It is these solutions that we will consider in the present workshop.

The KdV equation has the form

$$\frac{\partial h}{\partial t} + \frac{\partial^3 h}{\partial x^3} + 6h\frac{\partial h}{\partial x} = 0. \tag{1}$$

Here $h(x,t) \in \mathbb{R}$ is the height of a shallow water wave at position $x \in \mathbb{R}$ and time $t \in \mathbb{R}$. The KdV equation is a partial differential equation, containing derivatives w.r.t. the two variables x and t. These derivatives are denoted by $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial t}$. Such derivatives will be covered later in ODEs, Curves and Dynamics. However here we are only interested in solutions of the form

$$h(x,t) = f(x - vt)$$

where v is a constant real number and f is a function of a single variable $s = x - vt$. In this case one can show that

$$\begin{aligned} \frac{\partial h}{\partial x} &= f'(x - vt) \\ \frac{\partial h}{\partial t} &= -vf'(x - vt) \end{aligned}$$

and hence Eq. (1) turns into

$$-vf'(s) + f'''(s) + 6f(s)f'(s) = 0. \tag{2}$$

You can take this formula as given when solving this workshop.

Recall:

- Let us recall how to solve a differential equation by separation, e.g.

  $$y'(x) = x^2 y(x)^2$$

  or, in different notation,

  $$\frac{dy}{dx} = x^2 y^2.$$

  We separate the terms depending on x and y, and then integrate on both sides, as in

  $$\int \frac{1}{y^2} dy = \int x^2 dx.$$

  Evaluating the integrals gives

  $$-\frac{1}{y} = \frac{1}{3}x^3 + A$$

  where A is a constant.

- The hyperbolic sine and cosine have the derivatives

$$\sinh'(x) = \cosh(x), \quad \cosh'(x) = \sinh(x)$$

and they are related by $\cosh^2(x) - \sinh^2(x) = 1$.

For further exploration:

(1) You can find an experimental realisation of solitons described by the KdV equation by searching for "Collision of KdV solitons" on Youtube. This video also demonstrates that solitons can move to the right or to the left, be reflected, and move through each other when they collide.

How would you get started with obtaining left moving solitons from the KdV equation? Just give the idea, you don't have to work out any details.

A different example for a solitary wave (whose underlying Maths is more complicated) is the Severn bore, a particular type of water wave that occurs regularly in the River Severn near Bristol.

(2) Determine the solutions for the case that the sign in Eq. (**??**) is chosen negative, as in

$$f'(s) = -f(s)\sqrt{v - 2f(s)}.$$

Show that they can be brought to the same form as for the case with a positive sign.

(3) If you had not been given the substitution (**??**), how could you have guessed that it is helpful to compute the integral? Which other substitutions could you have tried?

(4) For which other differential equations can the method in 1(c) (multiply with $f'(s)$ and integrate) be helpful? Explain your answer. You don't need to give the most general type of such equations. Do you recall an example where this method was used to derive a Mechanics result in ODEs, Curves and Dynamics?