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CONSTRUCTING SIMULTANEOUS HECKE EIGENFORMS

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It is well known that newforms of integral weight are simultaneous eigenforms for all the Hecke operators, and that the converse is not true. In this paper, we give a characterization of all simultaneous Hecke eigenforms associated to a given newform, and provide several applications. These include determining the number of linearly independent simultaneous eigenforms in a fixed space which correspond to a given newform, and characterizing several situations in which the full space of cusp forms is spanned by a basis consisting of such eigenforms. Part of our results can be seen as a generalization of results of Choie–Kohnen who considered diagonalization of "bad" Hecke operators on spaces with square-free level and trivial character. Of independent interest, but used herein, is a lower bound for the dimension of the space of newforms with arbitrary character.

Keywords: Simultaneous Hecke eigenforms; newforms.

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1. Introduction

For N a positive integer, ψ a Dirichlet character defined modulo N, and $k \geq 2$ an integer, we let $S_k(N,\psi)$ denote the space of cusp forms of weight k for $\Gamma_0(N)$ with character ψ, and $S_k^+(N,\psi)$ the subspace generated by the newforms. For a prime p, we let T_p (or T_p^N) denote the pth Hecke operator for forms on $S_k(N,\psi)$. We use this notation for the primes which divide the level as well, so for example if $q \mid N$, our Hecke operator T_q^N is the same as the operator U_q in the notation of [3].

It is well known that $S_k(N, \psi)$ has a basis consisting of simultaneous eigenforms for the algebra of Hecke operators generated by $\{T_p^N \mid (p, N) = 1\}$, and via multiplicity-one that $S_k^+(N, \psi)$ has a basis of simultaneous eigenforms for all the Hecke operators. Since $S_k^+(N, \psi)$ is generally a proper subspace of $S_k(N, \psi)$, it is a natural question to consider the extent to which the full space of cusp forms has a basis of simultaneous eigenforms for all the Hecke operators. Choie and Kohnen [2] considered the question of diagonalizing "bad" Hecke operators (that is, T_q^N where $q \mid N$), and gave an upper bound for the number of primes q for which T_q^N could not be diagonalized on $S_k(N, \psi)$ where N is square-free, $q|N$ and ψ is trivial. An alternate perspective on that question is to determine conditions under which simultaneous Hecke eigenforms are newforms. One result along these lines is Li's [3] Theorem 9: if $f \in S_k(N, \psi)$ is a simultaneous eigenform for all Hecke operators T_p^N, and f is also an eigenform for the operator KW_N (where K is the conjugation operator and W_N is the Fricke involution), then f is a newform.

In this paper, we address the question broadly, in particular giving a characterization of all simultaneous Hecke eigenforms associated to a given newform for arbitrary level and character. For a given newform $h \in S_k(N_0, \psi)$, we first determine (Theorem 2.1 and Proposition 2.2) the exact structure and the eigenvalues of each form $f \in S_k(N, \psi)$ which is Hecke-equivalent to h and also an eigenfunction for T_q^N. In Sec. 3, we address the diagonalizability of T_q^N on a given space of cusp forms, characterizing several situations in which the full space of cusp forms is spanned by a basis consisting of such eigenforms, as well as those situations when it is not (Theorems 3.3 and 3.6). To establish the later result we derive a lower bound (Theorem 6.1) for the dimension of the space of newforms, $S_k^+(N, \psi)$; dimension formulas for the space of newforms with trivial character are given by Martin [4]. In Theorem 3.4, we generalize the results of Choie–Kohnen producing an upper bound for the number of primes q for which T_q^N fails to diagonalize. Section 4 considers simultaneous Hecke eigenforms, and Sec. 5 has several examples delineating cases in which bases of simultaneous eigenforms do or do not exist.

2. Characterizing Hecke Eigenforms at Primes Dividing the Level

Throughout, we make the convention that all Dirichlet characters will be considered as defined modulo their conductor, so that when considering a modular form in $S_k(N, \psi)$, $\psi(d) \neq 0$ iff d is relatively prime to the conductor. In particular, there may well be primes $q \mid N$ for which $\psi(q) \neq 0$. Of course for any prime $q \nmid N$, $\psi(q) \neq 0$. The convention is necessary to allow a uniform handling of all subspaces $S_k(N_0, \psi)$ where $\mathrm{cond}(\psi) \mid N_0 \mid N$.

Let $h \in S_k(N_0, \psi)$ be a newform (always assumed nonzero), N an integer divisible by N_0, and $f \in S_k(N, \psi)$ a nonzero simultaneous Hecke eigenform having the same eigenvalues as h for all Hecke operators T_ℓ^N, ℓ a prime with $(\ell, N) = 1$. The eigenvalues of h are given by $h \,|\, T_\ell^{N_0} = \lambda_\ell h$ for all primes ℓ, and we note that $T_\ell^N = T_\ell^{N_0}$ when $(\ell, N) = 1$. Moreover suppose that f is also a nonzero eigenform for T_q^N where q is a fixed prime dividing N, and put $f \,|\, T_q^N = \kappa_q f$. It is well known ([1, 3]), that f has the form

$$f = \sum_{d | \frac{N}{N_0}} \alpha_d h \,|\, B_d,$$

where B_d (also sometimes denoted V_d) is the shift operator of [3], and the α_d are complex scalars.

Theorem 2.1. *Let the notation be as above. Then assuming $q \,|\, N$ and $d \,|\, N/N_0$, we have*:

(1) *If $q \,|\, N_0$ then $\alpha_d = 0$ if $q^2 \,|\, d$.*

(a) *If $q \nmid N/N_0$, then $\kappa_q = \lambda_q$, and (vacuously) $\alpha_d = 0$ for $q \,|\, d$.*
(b) *If $q \,|\, N/N_0$, then $\alpha_d = (\kappa_q - \lambda_q)\alpha_{d/q}$ if $q \,\|\, d$. If $\kappa_q \neq 0$, then $\kappa_q = \lambda_q$, and $\lambda_q = 0$ implies $\kappa_q = 0$.*

(2) *If $q \nmid N_0$, then $\alpha_d = 0$ if $q^3 \,|\, d$.*

(a) *If $\kappa_q \neq 0$, then $\alpha_d = 0$ if $q^2 \,|\, d, \alpha_d = (\kappa_q - \lambda_q)\alpha_{d/q}$ if $q \,\|\, d$, and $\kappa_q = \frac{1}{2}(\lambda_q \pm \sqrt{\lambda_q^2 - 4\psi(q)q^{k-1}}) \neq \lambda_q$.*
(b) *If $\kappa_q = 0$, then $q^2 \,|\, N/N_0, \alpha_d = \psi(q)q^{k-1}\alpha_{d/q^2}$ if $q^2 \,\|\, d$, and $\alpha_d = -\lambda_q \alpha_{d/q}$ if $q \,\|\, d$.*

Proof. As above, we assume that $f = \sum_{d | \frac{N}{N_0}} \alpha_d h \,|\, B_d$. We separate the argument by cases.

- **Case:** $q \,|\, N_0, q \nmid N/N_0$.
  Since $q \,|\, N_0, T_q^N = T_q^{dN_0}$ for any d. Also note that since $q \nmid N/N_0$, any divisor $d \,|\, N/N_0$ satisfies $(d, q) = 1$, so that the shift and Hecke operators commute: $B_d \,|\, T_q^{dN_0} = T_q^{N_0} \,|\, B_d$. Thus

$$\kappa_q f = f \,|\, T_q^N = \sum_{d | \frac{N}{N_0}} \alpha_d h \,|\, B_d \,|\, T_q^{dN_0} = \sum_{d | \frac{N}{N_0}} \alpha_d h \,|\, T_q^{N_0} \,|\, B_d$$

$$= \lambda_q \sum_{d | \frac{N}{N_0}} \alpha_d h \,|\, B_d = \lambda_q f,$$

  and so we have $\kappa_q = \lambda_q$. Since $q \nmid N/N_0$, it is vacuously true that $\alpha_d = 0$ for $q \,|\, d$.

- **Case:** $q \mid N_0, q \mid N/N_0$.
  As in the previous case, we note that $q \mid N_0$ implies $T_q^N = T_q^{dN_0}$ for any d, and for any divisor $d \mid N/N_0$ satisfying $(d,q) = 1$, the shift and Hecke operators commute: $B_d \mid T_q^{dN_0} = T_q^{N_0} \mid B_d$. Finally we note that $B_q T_q^N = 1$. With these observations we have

$$\begin{aligned}\kappa_q f = f \mid T_q^N &= \sum_{d \mid N/N_0} \alpha_d h \mid B_d \mid T_q^{dN_0} \\ &= \sum_{d, q \nmid d} \alpha_d h \mid T_q^{N_0} \mid B_d + \sum_{d, q \mid d} \alpha_d h \mid B_{d/q} \\ &= \sum_{d, q \nmid d} (\lambda_q \alpha_d + \alpha_{dq}) h \mid B_d + \sum_{d, q^2 \mid d} \alpha_d h \mid B_{d/q}.\end{aligned}$$

  We now show the second summand does not appear.

Lemma. $\alpha_d = 0$ if $q^2 \mid d$.

Proof. If $\kappa_q = 0$, then the linear independence of $\{h \mid B_d\}$ yields the result. If $\kappa_q \neq 0$, let $M = \max_{d \mid N/N_0}\{\mathrm{ord}_q(d) \mid \alpha_d \neq 0\}$. Then we have

$$\begin{aligned}\kappa_q f = \sum_{d \mid N/N_0} \kappa_q \alpha_d h \mid B_d &= \sum_{i=0}^{M} \sum_{d \mid N/N_0, q^i \parallel d} \kappa_q \alpha_d h \mid B_d \\ &= \sum_{d, q \nmid d} (\lambda_q \alpha_d + \alpha_{dq}) h \mid B_d + \sum_{d, q^2 \mid d} \alpha_d h \mid B_{d/q}.\end{aligned}$$

There is no issue if $M < 2$, so we assume $M \geq 2$. In that case for a divisor d with $\mathrm{ord}_q(d) = M$, we see that a term with $h \mid B_d$ occurs as a summand in $\kappa_q f$, but not in $\sum_{d, q^2 \mid d} \alpha_d h \mid B_{d/q}$, so $\alpha_d = 0$, a contradiction. $\square$

Applying this observation, the equation above becomes:

$$\sum_{d \mid N/N_0, q^2 \nmid d} \kappa_q \alpha_d h \mid B_d = \kappa_q f = f \mid T_q^N = \sum_{d \mid N/N_0, q \nmid d} (\lambda_q \alpha_d + \alpha_{dq}) h \mid B_d. \quad (2.1)$$

By the linear independence of the set $\{h \mid B_d\}$, we deduce from Eq. (2.1) that $\kappa_q \alpha_d = 0$ when $q \parallel d$. If $\kappa_q = 0$, then Eq. (2.1) is zero, hence the coefficients of $h \mid B_d$ are all zero and we conclude

$$\alpha_d = -\lambda_q \alpha_{d/q} = (\kappa_q - \lambda_q) \alpha_{d/q} \quad \text{if } q \parallel d.$$

On the other hand, if $\kappa_q \neq 0$, then $\alpha_d = 0$ when $q \parallel d$, so Eq. (2.1) becomes

$$\kappa_q f = \sum_{d \mid N/N_0, q \nmid d} \kappa_q \alpha_d h \mid B_d = \sum_{d \mid N/N_0, q \nmid d} \lambda_q \alpha_d h \mid B_d = \lambda_q f,$$

and hence $\kappa_q = \lambda_q$. It follows that $\lambda_q = 0$ implies $\kappa_q = 0$, and $0 = \alpha_d = (\kappa_q - \lambda_q) \alpha_{d/q}$ if $q \parallel d$.

- **Case:** $q \nmid N_0, q \mid N/N_0$.
  In this case $T_q^N = T_q^{N_0} - \psi(q)q^{k-1}B_q$, and we have

$$\begin{aligned}
\kappa_q f &= \sum_{d|N/N_0} \kappa_q \alpha_d h \mid B_d = \sum_{d|N/N_0} \alpha_d h \mid B_d \mid T_q^N \\
&= \sum_{d, q\nmid d} \alpha_d h \mid T_q^N \mid B_d + \sum_{d, q|d} \alpha_d h \mid B_{d/q} \\
&= \sum_{d, q\nmid d} \alpha_d h \mid \left(T_q^{N_0} - \psi(q)q^{k-1}B_q\right) \mid B_d + \sum_{d, q|d} \alpha_d h \mid B_{d/q} \\
&= \sum_{d, q\nmid d} (\lambda_q \alpha_d + \alpha_{dq}) h \mid B_d + \sum_{d, q\|d} (\alpha_{dq} - \psi(q)q^{k-1}\alpha_{d/q}) h \mid B_d \\
&\quad + \sum_{d, q^3|d} \alpha_d h \mid B_{d/q}. \qquad (2.2)
\end{aligned}$$

To simplify this expression, we show

Lemma. *$\alpha_d = 0$ if $q^3 \mid d$.*

Proof. This is completely analogous to the previous lemma. Let $M = \max_{d|N/N_0}\{\text{ord}_q(d) \mid \alpha_d \neq 0\}$. There is no issue if $M < 3$, so assume $M \geq 3$. In that case for a divisor d with $\text{ord}_q(d) = M$, we see that a term with $h \mid B_d$ occurs in $\kappa_q f$, but not $\sum_{d, q^3|d} \alpha_d h \mid B_{d/q}$, so $\alpha_d = 0$, a contradiction. □

To go further, we first suppose that $\kappa_q \neq 0$. If $q \| N/N_0$, we have $\alpha_d = 0$ for $q^2 \mid d$ by convention. Otherwise, let $d \mid N/N_0$ with $q^2 \mid d$. The coefficient of $h \mid B_d$ in $\kappa_q f$ is $\kappa_q \alpha_d$ while it is α_{dq} in $\sum_{d, q^3|d} \alpha_d h \mid B_{d/q}$. By the lemma, $\alpha_{qd} = 0$, so we infer $\alpha_d = 0$.

Applying these observations to the above expression for $\kappa_q f$ yields

$$\begin{aligned}
\kappa_q f = \sum_{d|N/N_0} \kappa_q \alpha_d h \mid B_d &= \sum_{d|N/N_0, q\nmid d} (\lambda_q \alpha_d + \alpha_{dq}) h \mid B_d \\
&\quad + \sum_{d|N/N_0, q\nmid d} -\psi(q)q^{k-1}\alpha_d h \mid B_{dq}. \qquad (2.3)
\end{aligned}$$

Comparing coefficients of $h \mid B_d$ and $h \mid B_{dq}$ we obtain for $q \| d$:

$$\begin{aligned}
\alpha_d &= (\kappa_q - \lambda_q)\alpha_{d/q}, \quad \text{and} \\
\kappa_q \alpha_d &= -\psi(q)q^{k-1}\alpha_{d/q}.
\end{aligned}$$

Substituting the expression for α_d from the first equation into the second yields the quadratic $(\kappa_q^2 - \lambda_q \kappa_q + \psi(q)q^{k-1})\alpha_{d/q} = 0$. Note that $\alpha_{d/q} = 0$ for all d with

$q \parallel d$ would imply $\alpha_d = 0$ for all $d \mid N/N_0$, hence $f = 0$. Thus

$$\kappa_q = \frac{1}{2}\big(\lambda_q \pm \sqrt{\lambda_q^2 - 4\psi(q)q^{k-1}}\big),$$

and we note that $\kappa_q \neq \lambda_q$ since $\psi(q) \neq 0$.

Finally, we assume $\kappa_q = 0$. Then all the coefficients of the $h \mid B_d$ in Eq. (2.2) are zero, yielding $\alpha_d \lambda_q + \alpha_{dq} = 0$ for $q \nmid d$, and $\alpha_{dq} = \psi(q)q^{k-1}\alpha_{d/q}$ for $q \parallel d$. Note that if $q \parallel N/N_0$, by convention we would have $\alpha_{dq} = 0$ in the last equation, leading to $\alpha_{d/q} = 0$ and hence $\alpha_d = 0$ implying $f = 0$. Thus $\kappa_q = 0$ forces $q^2 \mid N/N_0$, which completes the proof. □

As above, let $h \in S_k(N_0, \psi)$ be a newform, and N an integer divisible by N_0. Denote the *class* of h by

$$[h] = \{f \in S_k(N, \psi) : f, h \text{ have the same eigenvalues for all } T_p^N, p \nmid N\}.$$

By the theory of newforms, we know

$$[h] = \bigoplus_{d \mid N/N_0} \langle h \mid B_d \rangle,$$

that is, $f \in [h]$ if and only if $f = \sum_{d \mid N/N_0} \alpha_d h \mid B_d$ for scalars α_d. It is clear from the general theory of newforms that any such f is a simultaneous eigenform for all Hecke operators T_p^N for primes $p \nmid N$. In Theorem 2.1, we have given necessary conditions on the coefficients α_d for f to be an eigenform for T_q^N for a prime $q \mid N$ and eigenvalue κ_q. However, the necessary conditions are also sufficient.

Proposition 2.2. *Let $h \in S_k(N_0, \psi)$ be a newform, N an integer divisible by N_0, and q a prime dividing N. Set $h \mid T_q^{N_0} = \lambda_q h$, and fix κ_q and constants α_d for $d \mid N/N_0$ according to the following scheme (any unconstrained constants are arbitrary):*

- *$q \mid N_0$ and $q \nmid N/N_0$: Let $\kappa_q = \lambda_q$.*
- *$q \mid N_0$ and $q \mid N/N_0$: Let $\kappa_q = \lambda_q$ or 0, and put $\alpha_d = 0$ if $q^2 \mid d$, and $\alpha_d = (\kappa_q - \lambda_q)\alpha_{d/q}$ if $q \parallel d$.*
- *$q \nmid N_0$: Set $\alpha_d = 0$ if $q^3 \mid d$.*

  (i) *Let $\kappa_q = \frac{1}{2}(\lambda_q \pm \sqrt{\lambda_q^2 - 4\psi(q)q^{k-1}})$, and note $\kappa_q \neq 0, \lambda_q$. For $q^2 \mid d$ put $\alpha_d = 0$; for $q \parallel d$, put $\alpha_d = (\kappa_q - \lambda_q)\alpha_{d/q}$.*

  (ii) *Moreover, if $q^2 \mid N$, we can also let $\kappa_q = 0$, and for $q^2 \parallel d$, put $\alpha_d = \psi(q)q^{k-1}\alpha_{d/q^2}$ and for $q \parallel d$, put $\alpha_d = -\lambda_q \alpha_{d/q} = (\kappa_q - \lambda_q)\alpha_{d/q}$.*

Then $f = \sum_{d \mid N/N_0} \alpha_d h \mid B_d$ is an eigenform for T_q^N with eigenvalue κ_q.

Proof. The proposition follows immediately from the computations already present in Theorem 2.1. □

3. Comparison to Choie–Kohnen

As in the previous section, let $h \in S_k(N_0, \psi)$ be a newform, and N an integer divisible by N_0, and denote the class of h by $[h]$. From [1, 3], we know that if $S_k^+(N_0, \psi)$ denotes the space generated by newforms of level N_0,

$$\begin{aligned} S_k(N, \psi) &= \bigoplus_{\mathrm{cond}(\psi) | N_0 | N} \bigoplus_{d | N/N_0} (S_k^+(N_0, \psi) \,|\, B_d) \\ &= \bigoplus_{\mathrm{cond}(\psi) | N_0 | N} \bigoplus_{h} [h], \end{aligned}$$

where the last sum is over normalized newforms $h \in S_k^+(N_0, \psi)$.

Lemma 3.1. *Let q be a prime dividing N. Then T_q^N is diagonalizable on $S_k(N, \psi)$ if and only if there is a basis of $S_k(N, \psi)$ consisting of simultaneous eigenforms for T_q^N as well as for all T_p^N, p a prime with $p \nmid N$. Moreover, for each $N_0 \,|\, N$ and each normalized newform $h \in S_k(N_0, \psi)$, T_q^N is diagonalizable on $S_k(N, \psi)$ if and only if it is diagonalizable on each class $[h]$.*

Proof. For both statements, only the forward direction requires proof. If T_q^N is diagonalizable on $S_k(N, \psi)$, then $S_k(N, \psi) = \bigoplus_i E_i$ where the E_i are the eigenspaces corresponding to the distinct eigenvalues of T_q^N. For a prime $p \nmid N$, the Hecke operators T_p^N and T_q^N commute so each eigenspace is invariant under all the $T_p^N, p \nmid N$. Since Hecke theory tells us that $S_k(N, \psi)$ admits a basis of simultaneous eigenforms for all the T_p^N, and each E_i is invariant under this collection of operators, each E_i also admits such a basis, $\mathcal{B}_i$, every element of which is also (by definition) an eigenform for T_q^N.

Now consider the second statement. Every element of the basis $\mathcal{B}_i$ belongs to a unique class $[h]$ of some newform $h \in S_k(N_0, \psi)$ with $N_0 \,|\, N$. We collect the elements of the $\mathcal{B}_i$ which belong to a given class $[h]$. Since $S_k(N, \psi)$ is the direct sum of such classes and all the $\mathcal{B}_i$ taken together span $S_k(N, \psi)$, we see that T_q^N is diagonalizable on each class $[h]$. □

Below we reverse the process of the lemma, starting with the class of a newform $[h]$, and investigate how to decompose the class $[h]$ into subspaces, extracting the various eigenspaces of T_q^N for $q \,|\, N$, and give conditions under which T_q^N can be diagonalized on $[h]$. We then use these results to generalize those of Choie and Kohnen [2]. We also apply these results in Sec. 4 to determine when there exist simultaneous eigenforms for all the Hecke operators, and determine the number of such eigenforms which are linearly independent.

For a prime $q \mid N$ and $h \in S_k(N_0, \psi)$ a newform, Theorem 2.1 implies that $[h]$ contains at most three eigenspaces for T_q^N. With $f \mid T_q^N = \kappa_q f$, we have

$$\kappa_q \in \begin{cases} \{\lambda_q\} & \text{when } q \nmid N/N_0, \\ \{0, \lambda_q\} & \text{when } q \mid N/N_0, \quad q \mid N_0, \\ \left\{\frac{1}{2}(\lambda_q \pm \sqrt{\lambda_q^2 - 4\psi(q)q^{k-1}})\right\} & \text{when } q \parallel N/N_0, \quad q \nmid N_0, \quad \text{and} \\ \left\{0, \frac{1}{2}(\lambda_q \pm \sqrt{\lambda_q^2 - 4\psi(q)q^{k-1}})\right\} & \text{when } q^2 \mid N/N_0, \quad q \nmid N_0. \end{cases} \tag{3.1}$$

When $q \nmid N/N_0$, we have observed (Proposition 2.2) that every element of $[h]$ is an eigenform for T_q^N having eigenvalue λ_q, so T_q^N diagonalizes on $[h]$. Thus we restrict our attention to the case where $q \mid N/N_0$. Write $N/N_0 = q^\mu M_0$, with $q \nmid M_0$. For $d_0 \mid M_0$, put $U_{d_0} = \bigoplus_{i=0}^{\mu} \langle h \mid B_{d_0 q^i} \rangle$ where $\langle h \mid B_d \rangle$ denotes the $\mathbb{C}$-linear span of $h \mid B_d$. Using that $[h] = \bigoplus_{d_0 \mid M_0} U_{d_0}$, Theorem 2.1 shows that every eigenform $f \in [h]$ with $f \mid T_q^n = \kappa_q f$ has the form $f = \sum_{d_0 \mid M_0} f_{d_0}$ with $f_{d_0} = \sum_{i=0}^{\mu} \alpha_{q^i d_0} h \mid B_{q^i d_0} \in U_{d_0}$, and Proposition 2.2 shows that each f_{d_0} also satisfies $f_{d_0} \mid T_q^N = \kappa_q f_{d_0}$. Thus T_q^N diagonalizes on $[h]$ if and only if it diagonalizes on each U_{d_0}. Further, Theorem 2.1 and Proposition 2.2 also show that each subspace U_{d_0} contains precisely m linearly independent eigenforms for T_q^N where m is the number of distinct eigenvalues κ_q given in Eq. (3.1). Since the dimension of $U_{d_0} = \mu + 1$, T_q^N diagonalizes on $[h]$ if and only if $m = \mu + 1$. Note that since $m \le 3$, T_q^N diagonalizes on $[h]$ only if $\mu \le 2$. Moreover when $\mu = 2$ and $q \mid N_0$, we see from above that there are at most $m = 2 < 3 = \mu + 1$ distinct eigenvalues, so once again T_q^N cannot diagonalize in this case.

We quantify the above observations a bit further. Still assuming $q \mid N/N_0$, if $q \mid N_0$, there are two distinct eigenvalues precisely when $\lambda_q \neq 0$; by [3, Theorem 3] this occurs if and only if $q \parallel N_0$ or $\mathrm{ord}_q(\mathrm{cond}(\psi)) = \mathrm{ord}_q(N_0)$. If $q \nmid N_0$, there are two independent eigenforms for T_q^N (with nonzero eigenvalues κ_q) precisely when $\lambda_q^2 \neq 4\psi(q)q^{k-1}$, that is when λ_q fails to achieve the Deligne bound. There is an additional independent eigenform with eigenvalue $\kappa_q = 0$ if and only if $\mu \ge 2$. For later convenience we denote by $\mathfrak{Q}_{N_0,h}$ the set of primes $q \mid N/N_0$ (just characterized) yielding a maximal number of distinct eigenvalues κ_q, and tabulate their number.

$\mu = \mathrm{ord}_q(N/N_0) \ge 1$; $q \in \mathfrak{Q}_{N_0,h}$ provided:		Number of distinct eigenvalues κ_q
$q \mid N_0$	$\mathrm{ord}_q(\mathrm{cond}(\psi)) = \mathrm{ord}_q(N_0)$ or $q \parallel N_0$	2
$q \nmid N_0$	$\lambda_q^2 \neq 4\psi(q)q^{k-1}$	$\min(3, \mu + 1)$

$\mu = \mathrm{ord}_q(N/N_0) \ge 1$; $q \notin \mathfrak{Q}_{N_0,h}$ provided:		Number of distinct eigenvalues κ_q
$q \mid N_0$	$q^2 \mid N_0$ and $q \mid N_0/\mathrm{cond}(\psi)$	1
$q \nmid N_0$	$\lambda_q^2 = 4\psi(q)q^{k-1}$	$\min(2, \mu)$

With this in hand, we now generalize the first part of Choie and Kohnen's theorem [2] characterizing when "bad" Hecke operators can be diagonalized.

Theorem 3.2. *For a prime* $q \mid N$, *the Hecke operator* T_q^N *is diagonalizable on* $S_k(N,\psi)$ *only if* $S_k(N,\psi)$ *contains no newform of level* N_0 *with* $q^3 \mid N/N_0$, *or with* $q^2 \mid N/N_0$ *and* $q \mid N_0$. *Assuming this condition,* T_q^N *is diagonalizable if and only if for each* N_0 *with* $\mathrm{cond}(\psi) \mid N_0 \mid N$ *and each newform* $h \in S_k(N_0,\psi)$ *with* $h \mid T_q^{N_0} = \lambda_q h$, *either* $q \nmid N/N_0$ *or* $q \in \mathfrak{Q}_{N_0,h}$.

Proof. We know that

$$S_k(N,\psi) = \bigoplus_{\mathrm{cond}(\psi)|N_0|N} \bigoplus_h [h],$$

where the sum is over normalized newforms $h \in S_k(N_0,\psi)$. By Lemma 3.1, it suffices to determine when T_q^N is diagonalizable on each class $[h]$. Given a newform $h \in S_k(N_0,\psi)$, we have seen from the discussion preceding the theorem that T_q^N is diagonalizable on $[h]$ only if $\mu = \mathrm{ord}_q(N/N_0) \le 2$ and if $\mu = 2$, $q \nmid N_0$. Thus the given conditions are necessary. Moreover, if $q \nmid N/N_0$, every element of $[h]$ is an eigenform for T_q^N, so we restrict our attention to the case $q \mid N/N_0$.

Consider a newform $h \in S_k(N_0,\psi)$. As before, write $N/N_0 = q^\mu M_0$, with $q \nmid M_0$, and recall we are assuming $\mu = 1$, or $\mu = 2$ and $q \nmid N_0$. For $d_0 \mid M_0$, put $U_{d_0} = \bigoplus_{i=0}^{\mu} \langle h \mid B_{d_0 q^i}\rangle$. We have observed above since $[h] = \bigoplus_{d_0|M_0} U_{d_0}$, that T_q^N diagonalizes on $[h]$ if and only if it diagonalizes on each U_{d_0}, and that T_q^N diagonalizes on U_{d_0} if and only if $\dim U_{d_0}$ is equal to the number of distinct eigenvalues κ_q. From the tables above it is clear that the dimension $(\mu+1)$ equals the number of distinct eigenvalues if and only if $q \in \mathfrak{Q}_{N_0,h}$. □

We summarize the above results in a more compact formulation.

Theorem 3.3. *Let* q *be prime, and let* ψ *be a Dirichlet character with conductor* $\mathfrak{f} = q^\nu M_0$, *with* $\nu \ge 0$ *and* $q \nmid M_0$. *Let* M *be an integer with* $M_0 \mid M$ *and* $q \nmid M$. *If* $s \le 2$, *then* T_q *is diagonalizable on* $S_k(q^{\nu+s}M,\psi)$ *if and only if one of the following is true*:

(1) $s = 0$,
(2) $s = 1$ *and* $\nu \ge 1$,
(3) $s > 0, \nu = 0$, *and* $S_k(q^{\nu+s}M,\psi)$ *contains no newform* h *of level* N_0 *with* $q \nmid N_0$, $T_q^{N_0}h = \lambda_q h$, *and* $\lambda_q^2 = 4\psi(q)q^{k-1}$, *or*
(4) $s = 2, \nu \ge 1$, *and* $S_k(q^{\nu+s}M,\psi)$ *contains no newform of level* N_0 *with* $\mathrm{ord}_q(N_0) = \nu$ *or* $\nu+1$.

Proof. Set $N = q^{\nu+s}M$. We first interpret Theorem 3.2 in this setting. Since $s \le 2$, $q^3 \nmid N/\mathfrak{f}$, so $S_k(N,\psi)$ contains no newform of level N_0 with $q^3|N/N_0$. The only case in which $\mathfrak{f} \mid N_0 \mid N$ with $q \mid N_0$ and $q^2 \mid N/N_0$ occurs when $s = 2$ and $\mathrm{ord}_q(N_0) = \nu \ge 1$. In this case, if $S_k(N,\psi)$ contains a newform of level N_0 then T_q^N is not diagonalizable. Otherwise, T_q^N is diagonalizable if and only if for each $\mathfrak{f} \mid N_0 \mid N$ and each newform $h \in S_k(N_0, \psi)$, either $q \nmid N/N_0$ or $q \in \mathfrak{Q}_{N_0,h}$.

First suppose that one of the conditions (1)–(4) hold. If (1) holds, then $q \nmid N/\mathfrak{f}$, so $q \nmid N/N_0$ for all $\mathfrak{f} \mid N_0 \mid N$, hence T_q^N is diagonalizable. If (2) holds, then $\mathrm{ord}_q(N_0) = \nu$ or $\nu+1$ for each $\mathfrak{f} \mid N_0 \mid N$. In the first case, $\mathrm{ord}_q(N_0) = \mathrm{ord}_q(\mathfrak{f})$ so $q \in \mathfrak{Q}_{N_0,h}$ for each h with such level. In the second case, $q \nmid N/N_0$. Therefore T_q^N is diagonalizable. Now suppose that (3) holds. Then $q \in \mathfrak{Q}_{N_0,h}$ for each newform h of level N_0 with $q \nmid N_0$. If $\mathrm{ord}_q(N_0) = s$ then $q \nmid N/N_0$. Finally if $s = 2$ and $q \parallel N_0$, then $q \in \mathfrak{Q}_{N_0,h}$ for each h with such level. Hence T_q^N is diagonalizable. Lastly, suppose that (4) holds. Then each newform h contained in $S_k(N, \psi)$ has level N_0 with $\mathrm{ord}_q(N_0) = \nu + 2$, so that $q \nmid N/N_0$. Therefore T_q^N is diagonalizable.

Now suppose that none of (1) through (4) is true. Then since (1) is false, $s = 1$ or 2. If $s = 1$, then since (2) is false, $\nu = 0$. Then since (3) is false, $S_k(N, \psi)$ must contain some newform h of level N_0 with $q \nmid N_0$ and $T_q^{N_0} h = \lambda_q h$ with $\lambda_q^2 = 4\psi(q)q^{k-1}$. Then $q \notin \mathfrak{Q}_{N_0,h}$, so T_q^N is not diagonalizable. Now suppose that $s = 2$. If $\nu = 0$ then by the previous argument, T_q^N is not diagonalizable. If $\nu \geq 1$ then since (4) is false, $S_k(N, \psi)$ must contain some newform h of level N_0 with either $\mathrm{ord}_q(N_0) = \nu$ or $\nu + 1$. If $\mathrm{ord}_q(N_0) = \nu$, then $q \mid N_0$ and $q^2 \mid N/N_0$ so T_q^N is not diagonalizable. If $\mathrm{ord}_q(N_0) = \nu + 1$ then $\mathfrak{f} \mid N_0/q$ and $q^2 \mid N_0$, so $q \notin \mathfrak{Q}_{N_0,h}$, and hence T_q^N is not diagonalizable. □

In the next result, we extend the work of Choie and Kohnen [2] (where they considered square-free level and trivial character) by showing that if k is even, $s = 1$ or 2 and $\nu = 0$, then Theorem 3.3(3) holds for all but finitely many primes q.

Theorem 3.4. *Let k be an even integer, and let ψ be a Dirichlet character whose conductor $\mathfrak{f}$ divides M. Then T_q is diagonalizable on both $S_k(qM, \psi)$ and $S_k(q^2M, \psi)$ for all primes $q \nmid M$ except for a finite number $r \leq C(M, k, \psi)$ of exceptions, where*

$$C(M, k, \psi) := \sum_{\mathrm{cond}(\psi) \mid M_0 \mid M} \dim S_k^+(M_0, \psi) \left(1 + \sum_{\mu \geq 1} \left[\frac{g_{M_0,k}}{2^\mu}\right]\right),$$

and

$$g_{M_0,k} = \sum_{\chi \bmod M_0} \dim S_k^+(M_0, \chi).$$

Proof. By Theorem 3.3, the only way that a given T_q can fail to diagonalize on either $S_k(qM, \psi)$ or $S_k(q^2M, \psi)$ is if there is a newform $h \in S_k(M_0, \psi)$ for some M_0 with $\mathfrak{f} \mid M_0 \mid M$ which has $T_q^{M_0} h = \lambda_q h$ with $\lambda_q^2 = 4\psi(q)q^{k-1}$. Fix an M_0 with $\mathfrak{f} \mid M_0 \mid M$ and a newform $h \in S_k(M_0, \psi)$ with eigenvalues λ_n. Let K_h be the field obtained by adjoining all the λ_n to $\mathbb{Q}$. It is known ([6, Proposition 2.8]) that K_h is a number field and contains the Nth roots of unity which arise as values of ψ. Let ζ be a primitive $2N$th root of unity, so that $\mathbb{Q}(\zeta^2) \subset K_h$ and hence $K_h(\zeta)/K_h$ is at most a quadratic extension. Since k is even, $\sqrt{q} \in K_h(\zeta)$ for each prime $q \nmid M$ such that

$\lambda_q^2 = 4\psi(q)q^{k-1}$. We call such a q an exceptional prime for h. Now if $p_1, p_2, \ldots, p_s$ are different primes, the degree of $\mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_s})/\mathbb{Q}$ is 2^s. Since

$$\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{q} : q \text{ an exceptional prime for } h) \subseteq K_h(\zeta)$$

and K_h is a finite extension of $\mathbb{Q}$, there must be a finite number r_h of exceptional primes for h. In particular, $r_h \leq \operatorname{ord}_2([K_h : \mathbb{Q}]) + 1$.

The group $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on normalized eigenforms in $S_k(M_0, \psi)$ by sending $f = \sum a(n)q^n \in S_k(M_0, \psi)$ to $f^\sigma = \sum a(n)^\sigma q^n \in S_k(M_0, \psi^\sigma)$, for each $\sigma \in \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ ([6, Proposition 2.6]). Let $K_{M_0,k} = \prod K_h$ be the composite field where the product runs over all characters χ modulo M_0 and all newforms $h \in S_k(M_0, \chi)$. Since each automorphism of the Galois closure of $K_{M_0,k}/\mathbb{Q}$ permutes these newforms, it can be considered as a subgroup of $S_{g_{M_0,k}}$, the symmetric group on $g_{M_0,k}$ elements, where

$$g_{M_0,k} = \sum_{\chi \bmod M_0} \dim S_k^+(M_0, \chi).$$

Then $[K_h : \mathbb{Q}] \,|\, [K_{M_0,k} : \mathbb{Q}] \,|\, g_{M_0,k}!$, so

$$r_h \leq 1 + \operatorname{ord}_2(g_{M_0,k}!) = 1 + \sum_{\mu \geq 1} \left[\frac{g_{M_0,k}}{2^\mu}\right].$$

Now T_q diagonalizes on neither $S_k(qM, \psi)$ nor $S_k(q^2M, \psi)$ if q is an exceptional prime for a single newform $h \in S_k(M_0, \psi)$ for any $\mathfrak{f} \,|\, M_0 \,|\, M$. Therefore we get an upper bound for r, the number of primes q for which T_q fails to diagonalize, by summing over all such newforms. Then

$$\begin{aligned} r &\leq \sum_{\operatorname{cond}(\psi)|M_0|M} \dim S_k^+(M_0, \psi) \cdot r_h \\ &\leq \sum_{\operatorname{cond}(\psi)|M_0|M} \dim S_k^+(M_0, \psi) \left(1 + \sum_{\mu \geq 1} \left[\frac{g_{M_0,k}}{2^\mu}\right]\right). \end{aligned}$$ □

Remark 3.5. One could obtain a more explicit, though considerably larger, upper bound. For example, $g_{M_0,k} \leq \dim S_k(\Gamma_1(M_0))$ for which one could use the known dimension formulas.

We conclude this investigation of diagonalization with the following "negative" result for levels divisible by a high power of q.

Theorem 3.6. *Let q be prime, and let ψ be a Dirichlet character with conductor $\mathfrak{f} = q^\nu M_0$, with $\nu \geq 0$ and $q \nmid M_0$. Let M be an integer with $M_0 \,|\, M$ and $q \nmid M$, and let $s \geq 3$ be an integer. Except possibly for finitely many $k \geq 2$ with $\psi(-1) = (-1)^k$ and finitely many q, T_q is not diagonalizable on $S_k(q^{\nu+s}M, \psi)$.*

Proof. Let $N = q^{\nu+s}M$ and $N_0 = q^{\nu+2}M$. For each $s \geq 3$, we have $\mathfrak{f} \,|\, N_0 \,|\, N$. Further, $q \,|\, N/N_0, q^2 \,|\, N_0$ and $\mathfrak{f} \,|\, N_0/q$. Hence if $S_k(N, \psi)$ contains a newform h of level N_0, then $q \notin \mathfrak{Q}_{N_0,h}$, so T_q^N is not diagonalizable. But by Theorem 6.1 (see Sec. 6), for all but finitely many $k \geq 2$ with $\psi(-1) = (-1)^k$ and finitely many q, $\dim S_k^+(N_0, \psi) \geq 1$, and hence T_q is not diagonalizable on $S_k(N, \psi)$. □

4. Simultaneous Hecke Eigenforms

We now turn to the question of characterizing simultaneous Hecke eigenforms in $S_k(N, \psi)$ for all Hecke operators T_ℓ^N, ℓ a prime. From the previous section and the theory of newforms, for a given simultaneous eigenform $f \in S_k(N, \psi)$, the only primes which need careful analysis are primes $q \mid N/N_0$ where N_0 is the level of the associated newform. We make this explicit.

Theorem 4.1. *Let $f \in S_k(N, \psi)$ be a nonzero simultaneous eigenform for all the Hecke operators T_ℓ^N, ℓ a prime, and put $f \mid T_q^N = \kappa_q f$ for each prime $q \mid N$. Associated to f is a newform $h \in S_k(N_0, \psi)$ (with $\mathrm{cond}(\psi) \mid N_0 \mid N$) such that $f = \sum_{d \mid N/N_0} \alpha_d h \mid B_d$. As before, put $h \mid T_q^{N_0} = \lambda_q h$. Then $\alpha_1 \neq 0$, and normalizing with $\alpha_1 = 1$, we have that $\alpha_d = \prod_{q \mid d} \alpha_{q^{\mu_q}}$, where $\mu_q = \mathrm{ord}_q(d)$. Further, we have $\alpha_{q^e} = 0$ for $e \geq 3$, and*

$$\alpha_q = (\kappa_q - \lambda_q) \quad \textit{and} \quad \alpha_{q^2} = \begin{cases} 0 & q \mid N_0, \\ 0 & q \nmid N_0, \quad \kappa_q \neq 0, \\ \psi(q)q^{k-1} & q \nmid N_0, \quad \kappa_q = 0. \end{cases}$$

Proof. This is immediate from Theorem 2.1, which also indicates the possible eigenvalues κ_q. □

Remark 4.2. The converse to the above theorem is also true. Starting with a newform h, and choosing the κ_q and α_d as in the theorem, Proposition 2.2 guarantees that $f = \sum_{d \mid M} \alpha_d h \mid B_d$ is a simultaneous eigenform for all T_q^N with $q \mid N$, and hence for all T_ℓ^N, ℓ a prime.

Now we wish to count the number of linearly independent simultaneous Hecke eigenforms that are associated to a given newform.

Theorem 4.3. *Let $h \in S_k(N_0, \psi)$ be a newform and let N be an integer such that $N_0 \mid N$. For all primes $q \mid N$, put $h \mid T_q^{N_0} = \lambda_q h$. The number of linearly independent simultaneous eigenforms $f \in S_k(N, \psi)$ which are eigenforms for all $\{T_\ell^N\}, \ell$ a prime and which have the same eigenvalues as h under all $T_p, p \nmid N$ is $2^{|A|}3^{|B|}$, where A and B are sets of primes dividing N/N_0 satisfying*

$$q \in B = B(N, N_0, h) \Leftrightarrow q \nmid N_0, q^2 \mid N/N_0, \lambda_q^2 \neq 4\psi(q)q^{k-1},$$

and

$$q \in A = A(N, N_0, h) \Leftrightarrow \begin{cases} q \mid N_0 \quad \textit{and} \quad \lambda_q \neq 0, \quad \textit{or} \\ q \nmid N_0, \quad q \parallel N/N_0, \quad \lambda_q^2 \neq 4\psi(q)q^{k-1}, \quad \textit{or} \\ q \nmid N_0, \quad q^2 \mid N/N_0, \quad \lambda_q^2 = 4\psi(q)q^{k-1}. \end{cases}$$

Remark 4.4. By [3, Theorem 3], the first condition stated to define A ($q \mid N_0$ and $\lambda_q \neq 0$) is equivalent to $q \parallel N_0$, or $q^2 \mid N_0$ and $\text{ord}_q(\text{cond}(\psi)) = \text{ord}_q(N_0)$.

Proof. Theorem 4.1 indicates the shape of every simultaneous eigenform f of level N associated to the newform $h : f = \sum_{d|N/N_0} \alpha_d h \mid B_d$, where without loss, $\alpha_1 = 1$, and α_d is completely determined as the product of α_{q^e} where $e = \text{ord}_q(d)$. We see all such values α_{q^e} are uniquely determined except for the value of $\alpha_q = \kappa_q - \lambda_q$ which has as many distinct values as distinct eigenvalues κ_q. It is now a simple matter using Theorem 2.1 to verify that the sets A and B characterize those cases in which κ_q can have two or three distinct eigenvalues. □

5. Examples

Theorem 4.3 tells how to compute the number of simultaneous eigenforms in $S_k(N, \psi)$ associated to a newform $h \in S_k(N_0, \psi)$ with $h \mid T_q^{N_0} = \lambda_q h$ in terms of the sets A and B. Knowledge of the eigenvalue λ_q for $q \nmid N_0$ can often be problematic, but there are cases in which it is easy to calculate explicitly the sets A and B. We characterize one particularly useful situation, and give some examples.

Let $N_0 \mid N$ with N and N_0 having exactly the same prime divisors. Then $B = B(N, N_0, h) = \emptyset$, and by Remark 4.4

$$A = A(N, N_0, h) = \{q \mid N/N_0 : \text{ord}_q(N_0) = 1 \text{ or } \text{ord}_q(\text{cond}(\psi))\}. \tag{5.1}$$

Example 5.1. Let ψ be a character with square-free conductor D, and let N be an integer with $D \mid N \mid D^2$. Then $S_k(N, \psi)$ has a basis consisting of simultaneous eigenforms for all Hecke operators.

Proof. Let N_0 be such that $D \mid N_0 \mid N$, and consider a newform $h \in S_k(N_0, \psi)$. To compute A, let $q \mid N/N_0$ be a prime. Observe that $q \mid N/N_0$ implies $q \mid D$. Since $N \mid D^2$, $1 \leq \text{ord}_q(N_0) \leq \text{ord}_q(N) \leq 2$, and if $\text{ord}_q(N_0) = 2$, then $q \nmid N/N_0$. Thus $q \mid N/N_0$ implies $\text{ord}_q(N_0) = 1$ and hence $q \in A$. Thus $2^{|A|} = \sigma_0(N/N_0)$ (since N/N_0 is square-free), where $\sigma_0(m)$ is the number of positive divisors of m. Since

$$S_k(N, \psi) = \bigoplus_{D|N_0|N} \bigoplus_{d|N/N_0} S_k^+(N_0, \psi) \mid B_d \cong \bigoplus_{D|N_0|N} \sigma_0(N/N_0) S_k^+(N_0, \psi),$$

with the isomorphism as modules for the Hecke algebra generated by T_p^N for all primes $p \nmid N$, the result is clear. In the isomorphism we use the convention that for a space $S, mS = \bigoplus_{i=1}^m S$. □

As a second example, we consider a situation in which the conductor of the character ψ can be large.

Example 5.2. Let q be an odd prime and ψ a character of conductor $q^\nu, \nu \geq 1$. Then $S_k(q^{\nu+\mu}, \psi), \mu \geq 0$, has a basis of simultaneous eigenforms for all the Hecke operators when $\mu = 0, 1$; for $\mu \geq 3$ it has such a basis only for finitely many k and q.

Proof. The only issue concerns the diagonalizability of the operator T_q^N where $N = q^{\nu+\mu}$. The case of $\mu = 0, 1$ is addressed by Theorem 3.3, while the case of $\mu \geq 3$ is addressed by Theorem 3.6. □

Example 5.3. Somewhat complementary to the previous example, we consider the case of $S_k(q^2, 1)$ providing instances when T_q^N can be diagonalized. To that end, we consider normalized newforms h of level 1, q and q^2 and whether T_q^N can be diagonalized on $[h]$. For level q^2 the answer is affirmative from the theory of newforms. For each newform $h \in S_k(q, 1)$, Eq. (5.1) yields $A = \{q\}$ providing the requisite two linearly independent simultaneous eigenforms, h and $h - \lambda_q h \,|\, B_q$. Now consider $h \in S_k(1, 1)$. When $k < 12$ or $k = 14$, $S_k(1, 1) = 0$, and there are no classes to consider. On the other hand, when $S_k(1, 1) \neq 0$, the situation is more subtle. By Theorem 3.3(iii), T_q^N will diagonalize on $[h]$ provided that $\lambda_q^2 \neq 4q^{k-1}$. As an example, consider weight 12. There are three simultaneous eigenforms in $S_{12}(q^2, 1)$ equivalent to $\Delta = (2\pi)^{12} \sum_{n=1}^{\infty} \tau(n) e^{2\pi i n z} \in S_{12}(1, 1)$ with Fourier coefficients given by the Ramanujan τ-function. Choose a prime q for which $\lambda_q^2 = \tau^2(q) \neq 4q^{11}$; note that this is true for all primes q since $\tau(q) \in \mathbb{Z}$ while $2q^{11/2}$ is not.

Using Theorem 4.1, we produce three linearly independent simultaneous eigenforms $f = \Delta + \alpha_q \Delta \,|\, B_q + \alpha_{q^2} \Delta \,|\, B_{q^2}$ satisfying $f \,|\, T_q^{q^2} = \kappa_q f$ where $\kappa_q = 0$, $\frac{1}{2}(\lambda_q \pm \sqrt{\lambda_q^2 - 4q^{11}})$ (all distinct). Thus T_q diagonalizes on $S_{12}(q^2, 1)$.

Example 5.4. The results of Sec. 2 can also provide a multiplicity-one theorem in the following narrow context. Let $N_0 \geq 2$ be an integer, and consider any newform $h \in S_k(N_0^2, 1)$. Let $N = N_0^2 N_1$ where any prime dividing N_1 also divides N_0. Let $f \in S_k(N, 1)$ be a simultaneous eigenform for all Hecke operators T_p for $p \nmid N$ which is equivalent to h. Then $f = h$.

Proof. This is immediate from Proposition 2.2, since for each $q \,|\, N$, the qth eigenvalue of $h(\lambda_q)$ is zero forcing $f = \sum \alpha_d h \,|\, B_d = h$. □

Remark 5.5. Another point worth noting concerns the interpretation when the sets A and B are empty. In such a case, Theorem 4.3 implies there is a unique simultaneous eigenform f in $S_k(N, \psi)$ associated to a newform $h \in S_k(N_0, \psi)$. It is not necessarily the case that the simultaneous eigenform is the newform h. For example, choose a prime $q \,\|\, N/N_0$ with $q \nmid N_0$. Then $B = \emptyset$ and $q \notin A$ means that $\lambda_q^2 = 4\psi(q) q^{k-1} \neq 0$ and hence by Theorem 4.1, $f = \sum_{d|N/N_0} \alpha_d h \,|\, B_d$ with $\alpha_q = -\lambda_q/2 \neq 0$.

6. Dimensions of Spaces of Newforms

To justify the last part of Theorem 3.3, we compute a lower bound for the dimension of the space of newforms $S_k^+(q^{\nu+r}, \psi)$ where q is a prime, $r \geq 2$, and ψ a character with conductor $q^\nu, \nu \geq 0$. We make implicit use of the trace formula for Hecke operators as given in [5], in particular Ross's formula for the dimension of the space of cusp forms. For trivial character, one can find a formula for the dimension of the space of newforms in [4].

Theorem 6.1. *Let q be a prime, and ψ be a Dirichlet character of conductor $\mathfrak{f} = q^{\nu} M_0, q \nmid M_0, \nu \geq 0$. Let r be an integer, $r \geq 2$, and let M be any integer divisible by M_0 $(q \nmid M)$; we further require that $\mathrm{ord}_2(M/M_0) \neq 1$. Then except for finitely many values of $k \geq 2$ with $\psi(-1) = (-1)^k$ and finitely many values of q, we have the dimension of the space of newforms $S_k^+(q^{\nu+r}M, \psi)$ is positive.*

Proof. We consider the Hecke algebra generated by all operators T_p with $p \nmid qM$, and recall our shorthand of writing mS for $\bigoplus_{i=1}^{m} S$ in any isomorphism of modules for the Hecke algebra. Let $N = q^{\nu+r}M$. Then

$$\begin{aligned} S_k(N, \psi) &= \bigoplus_{\mathfrak{f} | N_0 | N} \bigoplus_{d | N/N_0} S_k^+(N_0, \psi) \,|\, B_d \\ &\cong \bigoplus_{\mathfrak{f} | N_0 | N} \sigma_0(N/N_0) S_k^+(N_0, \psi) \\ &= \bigoplus_{d | N/\mathfrak{f}} \sigma_0(N/d\mathfrak{f}) S_k^+(d\mathfrak{f}, \psi). \end{aligned}$$

We adapt the notation of Martin [4], who gives a formula for the dimension of the space of newforms with trivial character, and put

$$g_0(d) = \dim S_k(d\mathfrak{f}, \psi), \quad g_0^+(d) = \dim S_k^+(d\mathfrak{f}, \psi).$$

The above decomposition yields the following relations of dimensions:

$$g_0(N/\mathfrak{f}) = \sum_{d | N/\mathfrak{f}} g_0^+(d) \sigma_0(N/d\mathfrak{f}) = (g_0^+ * \sigma_0)(N/\mathfrak{f}), \tag{6.1}$$

where $*$ is the standard Dirichlet convolution of arithmetic functions. While $g_0^+(n)$ is not a multiplicative function, $\sigma_0(n)$ is, and in complete analogy with [4], we let λ be the Dirichlet inverse of $\sigma_0 : \lambda = \mu * \mu$ (μ the Möbius function). Thus λ is a multiplicative function with values (for p a prime) $\lambda(p) = -2, \lambda(p^2) = 1$, and $\lambda(p^j) = 0$ for $j \geq 3$. Taking the Dirichlet convolution of both sides of Eq. (6.1) yields

$$\dim S_k^+(N, \psi) = g_0^+(N/\mathfrak{f}) = \sum_{d | N/\mathfrak{f}} g_0(d)\, \lambda(N/d\mathfrak{f}) = (g_0 * \lambda)(N/\mathfrak{f}). \tag{6.2}$$

The goal is to use the above expression to produce a formula for the dimension of the space of newforms as a function of the prime q and weight k. Using a parametrized version of the notation from [5], we obtain a formula for the dimension of the space of cusp forms with arbitrary character:

$$g_0(d) = -s_0(d\mathfrak{f}) - s_1(d\mathfrak{f}) + \delta + m(d\mathfrak{f}) - p(d\mathfrak{f}), \tag{6.3}$$

where letting $\omega(n)$ be the number of prime divisors of an integer n, we have

$$|s_0(d\mathfrak{f})| \le 2^{\omega(d\mathfrak{f})-2},$$

$$|s_1(d\mathfrak{f})| \le \frac{1}{3}2^{\omega(d\mathfrak{f})},$$

$$\delta = \begin{cases} 1 & \text{if } k = 2 \quad \text{and} \quad \psi = 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$m(d\mathfrak{f}) = \frac{(k-1)}{12} d\mathfrak{f} \prod_{\ell | d\mathfrak{f}} (1 + 1/\ell) \quad (\ell \text{ a prime}), \quad \text{and}$$

$$p(d\mathfrak{f}) = \frac{1}{2} \prod_{\ell | d\mathfrak{f}} \mathrm{par}(\ell) \quad (\ell \text{ a prime}),$$

where $\mathrm{par}(\ell)$ are the parabolic terms as computed in [5, Theorem 1].

Now we distribute the convolution of λ through the summands defining g_0 and obtain

$$\begin{aligned} g_0^+(N/\mathfrak{f}) &= \sum_{d|N/\mathfrak{f}} g_0(d)\lambda(N/d\mathfrak{f}) \\ &= -\sum_{d|N/\mathfrak{f}} s_0(d\mathfrak{f})\lambda(N/d\mathfrak{f}) - \sum_{d|N/\mathfrak{f}} s_1(d\mathfrak{f})\lambda(N/d\mathfrak{f}) + \delta \sum_{d|N/\mathfrak{f}} \lambda(N/d\mathfrak{f}) \\ &\quad + \sum_{d|N/\mathfrak{f}} m(d\mathfrak{f})\,\lambda(N/d\mathfrak{f}) - \sum_{d|N/\mathfrak{f}} p(d\mathfrak{f})\,\lambda(N/d\mathfrak{f}). \end{aligned} \tag{6.4}$$

If we put $N/\mathfrak{f} = q^r M_1$ with $q \nmid M_1$, then since $r \ge 2$,

$$\sum_{d|N/\mathfrak{f}} \lambda(N/d\mathfrak{f}) = \sum_{j=0}^{r} \lambda(q^j) \sum_{d_1|M_1} \lambda(M_1/d_1) = 0,$$

so

$$\begin{aligned} g_0^+(N/\mathfrak{f}) &= \sum_{d|N/\mathfrak{f}} g_0(d)\lambda(N/d\mathfrak{f}) \\ &= -\sum_{d|N/\mathfrak{f}} (s_0(d\mathfrak{f}) + s_1(d\mathfrak{f}))\lambda(N/d\mathfrak{f}) + \sum_{d|N/\mathfrak{f}} (m(d\mathfrak{f}) - p(d\mathfrak{f}))\lambda(N/d\mathfrak{f}), \end{aligned} \tag{6.5}$$

hence

$$\begin{aligned} g_0^+(N/\mathfrak{f}) &= |g_0^+(N/\mathfrak{f})| \\ &\ge \left| \sum_{d|N/\mathfrak{f}} (m(d\mathfrak{f}) - p(d\mathfrak{f}))\lambda(N/d\mathfrak{f}) \right| - \left| \sum_{d|N/\mathfrak{f}} (s_0(d\mathfrak{f}) + s_1(d\mathfrak{f}))\lambda(N/d\mathfrak{f}) \right|. \end{aligned} \tag{6.6}$$

We shall show that the second term is bounded and the first goes to infinity as q or k do which will establish our result.

We consider the second term:

$$\left|\sum_{d|N/\mathfrak{f}} (s_0(d\mathfrak{f}) + s_1(d\mathfrak{f}))\lambda(N/d\mathfrak{f})\right| \leq \sum_{d|N/\mathfrak{f}} (|s_0(d\mathfrak{f})| + |s_1(d\mathfrak{f})|)|\lambda(N/d\mathfrak{f})|$$
$$\leq \sum_{d|N/\mathfrak{f}} 2^{\omega(d\mathfrak{f})}|\lambda(N/d\mathfrak{f})|, \tag{6.7}$$

since from above we have that

$$|s_0(d\mathfrak{f})| + |s_1(d\mathfrak{f})| \leq 2^{\omega(d\mathfrak{f})-2} + \frac{1}{3}2^{\omega(d\mathfrak{f})} < 2^{\omega(d\mathfrak{f})}. \tag{6.8}$$

Writing $N/\mathfrak{f} = q^r M_1$ with $q \nmid M_1$ and recalling that $\mathfrak{f} = q^\nu M_0$, we have

$$\sum_{d|N/\mathfrak{f}} 2^{\omega(d\mathfrak{f})}|\lambda(N/d\mathfrak{f})| = \sum_{j=0}^{r} |\lambda(q^{r-j})| \sum_{d_1|M_1} 2^{\omega(q^{j+\nu}d_1M_0)}|\lambda(M_1/d_1)|$$
$$\leq \sum_{j=0}^{r} |\lambda(q^{r-j})| \sum_{d_1|M_1} 2^{1+\omega(M)}|\lambda(M_1/d_1)|$$
$$\leq 2^{3+\omega(M)} \sum_{d_1|M_1} |\lambda(M_1/d_1)|,$$

which is a constant depending only on M and independent from k and q. Thus it remains only to show that

$$\left|\sum_{d|N/\mathfrak{f}} (m(d\mathfrak{f}) - p(d\mathfrak{f}))\lambda(N/d\mathfrak{f})\right| \to \infty,$$

as q or k go to infinity. What we show is that

$$\sum_{d|N/\mathfrak{f}} (m(d\mathfrak{f}) - p(d\mathfrak{f}))\lambda(N/d\mathfrak{f}) = \frac{k-1}{12}A\mathcal{M}(q) - B\mathcal{P}(q),$$

with constants A, B depending only on M, $A > 0$, and $\mathcal{M}, \mathcal{P}$ functions of q with the expression having the desired limits as q or k go to infinity.

We first consider the "Mass" term: $\sum_{d|N/\mathfrak{f}} m(d\mathfrak{f})\lambda(N/d\mathfrak{f})$. For an integer n, let $m_0(n) = n\prod_{\ell|n}(1 + 1/\ell)$ where the product is over all primes ℓ dividing n. Then $m(n) = \frac{k-1}{12}m_0(n)$, and m_0 is a multiplicative function. Thus once again writing $N/\mathfrak{f} = q^r M_1$ with $q \nmid M_1$ and recalling that $\mathfrak{f} = q^\nu M_0$,

$$\sum_{d|N/\mathfrak{f}} m(d\mathfrak{f})\,\lambda(N/d\mathfrak{f}) = \frac{k-1}{12}\sum_{j=0}^{r} m_0(q^{\nu+j})\lambda(q^{r-j}) \sum_{d_1|M_1} m_0(d_1M_0)\lambda(M_1/d_1)$$

$$\begin{aligned}&= \frac{k-1}{12}(m_0(q^{\nu+r}) - 2m_0(q^{\nu+r-1}) + m_0(q^{\nu+r-2}))\\ &\quad\times \sum_{d_1\,|\,M_1} m_0(d_1M_0)\lambda(M_1/d_1)\\ &= \frac{k-1}{12}\mathcal{M}(q)\sum_{d_1\,|\,M_1} m_0(d_1M_0)\lambda(M_1/d_1),\end{aligned}$$

where

$$\mathcal{M}(q) = \begin{cases} q^{\nu+r-2}\left(1+\dfrac{1}{q}\right)(q-1)^2 & \text{if } \nu+r-2>0,\\ q^2-q-1 = (q-1)^2+q-2 & \text{if } \nu=0, r=2.\end{cases}$$

We now wish to show that $\sum_{d_1|M_1} m_0(d_1M_0)\lambda(M_1/d_1)$ is positive. Since both m_0 and λ are multiplicative, it suffices to show this when $M_1 = p^e, M_0 = p^f$ are prime powers ($e+f\geq 1$).

$$\begin{aligned}&\sum_{d_1\,|\,M_1} m_0(d_1M_0)\lambda(M_1/d_1)\\ &\qquad= \sum_{j=0}^{e} m_0(p^{f+j})\lambda(p^{e-j})\\ &\qquad= \begin{cases} m_0(p^f) & \text{if } e=0\\ -2m_0(p^f)+m_0(p^{f+1}) & \text{if } e=1\\ m_0(p^{e-2+f}) - 2m_0(p^{e-1+f}) + m_0(p^{e+f}) & \text{if } e\geq 2\end{cases}\\ &\qquad= \begin{cases} p^f + p^{f-1} & \text{if } e=0\\ p^{f+1}-p^f-2p^{f-1} & \text{if } e=1\\ p^2-p-1 & \text{if } e=2, f=0\\ p^{e+f-3}(p+1)(p-1)^2 & \text{if } e+f\geq 3,\end{cases}\end{aligned}$$

where we understand $p^{-1}=0$, and this sum is trivially checked to be positive for all primes $p\geq 2$. Note the case with $e=1$ (when $p=2$) is precluded by the theorem's hypothesis $\mathrm{ord}_2(M/M_0) = \mathrm{ord}_2(M_1)\neq 1$.

Finally we turn to the parabolic terms: $\sum_{d|N/\mathfrak{f}} p(d\mathfrak{f})\lambda(N/d\mathfrak{f})$. For an integer n, let $p_0(n) = \prod_{\ell|n}\mathrm{par}(\ell)$ where the product is over all primes ℓ dividing n, and $\mathrm{par}(\ell)$ is defined as in [5, Theorem 1]. Then $p(n) = (1/2)p_0(n)$, and p_0 is a multiplicative function, $p_0(1)=1$. Once again writing $N/\mathfrak{f} = q^rM_1$ with $q\nmid M_1$ and recalling that $\mathfrak{f} = q^\nu M_0$,

$$\sum_{d|N/\mathfrak{f}} p(d\mathfrak{f})\lambda(N/d\mathfrak{f}) = \frac{1}{2}\sum_{j=0}^{r} p_0(q^{\nu+j})\lambda(q^{r-j})\sum_{d_1\,|\,M_1} p_0(d_1M_0)\lambda(M_1/d_1)$$

$$= \frac{1}{2}(p_0(q^{\nu+r}) - 2p_0(q^{\nu+r-1}) + p_0(q^{\nu+r-2})) \times \sum_{d_1|M_1} p_0(d_1M_0)\lambda(M_1/d_1)$$

$$= \frac{1}{2}\mathcal{P}(q) \sum_{d_1|M_1} p_0(d_1M_0)\lambda(M_1/d_1),$$

where $\mathcal{P}(q) = (p_0(q^{\nu+r}) - 2p_0(q^{\nu+r-1}) + p_0(q^{\nu+r-2}))$.

Thus it remains only to show that

$$\left|\frac{k-1}{12}A\mathcal{M}(q) - B\mathcal{P}(q)\right| \to \infty, \quad \text{as } k \text{ or } q \to \infty$$

with constants A, B depending only on $M = M_0M_1$, $A > 0$. The case for $k \to \infty$ is clear, so we focus on this expression as a function of q.

To compute $p_0(q^{\nu+j})$, we set a bit of notation. Let $\mu_j = \lfloor \frac{\nu+j}{2} \rfloor$. From [5, Theorem 1], we have

$$p_0(q^{\nu+j}) = \mathrm{par}(q) = \begin{cases} 2q^j & \text{if } \nu \geq \mu_j + 1, \\ (q^{\mu_j} + q^{\mu_j - 1}) & \text{if } \nu \leq \mu_j, \quad \nu + j \text{ even}, \\ 2q^{\mu_j} & \text{if } \nu \leq \mu_j, \quad \nu + j \text{ odd}. \end{cases}$$

Since we need to compute $\mathcal{P}(q) = (p_0(q^{\nu+r}) - 2p_0(q^{\nu+r-1}) + p_0(q^{\nu+r-2}))$ as part of $|\frac{k-1}{12}A\mathcal{M}(q) - B\mathcal{P}(q)|$, we need to break the argument into cases.

Case 0. Special case $\nu = 0, r = 2$.

$$\left|\frac{k-1}{12}A\mathcal{M}(q) - B\mathcal{P}(q)\right| = \left|\frac{k-1}{12}A(q^2 - q - 1) - B(q-2)\right| \to \infty \quad \text{as } q \to \infty.$$

Henceforth we can assume $\nu + r \geq 3$, so $\mathcal{M}(q) = q^{\nu+r-2}\left(1 + \frac{1}{q}\right)(q-1)^2$.

Case 1. $\nu \geq \mu_r + 1$. Then $\nu \geq \mu_r + 1 \geq \mu_{r-1} + 1 \geq \mu_{r-2} + 1$. Note that this case cannot occur unless $\nu \geq 3$. Then $\mathcal{P}(q) = 2q^{r-2}(q-1)^2$, so

$$\left|\frac{k-1}{12}A\mathcal{M}(q) - B\mathcal{P}(q)\right|$$

$$= \left|\frac{k-1}{12}Aq^{\nu+r-2}\left(1 + \frac{1}{q}\right)(q-1)^2 - B(2q^{r-2}(q-1)^2)\right|$$

$$= q^{r-2}(q-1)^2\left|\frac{k-1}{12}Aq^{\nu}\left(1 + \frac{1}{q}\right) - B\right| \to \infty \quad \text{as } q \to \infty.$$

Case 2. $\boldsymbol{\nu \le \mu_{r-2}}$**.** Then $\nu \le \mu_{r-2} \le \mu_{r-1} \le \mu_r$.

When $\nu + r$ is even, $\mu_r = (\nu + r)/2 \ge 2$, and we have

$$\begin{aligned}\mathcal{P}(q) &= (q^{\mu_r} + q^{\mu_r - 1}) - 4q^{\mu_{r-1}} + q^{\mu_{r-2}} + q^{\mu_{r-2}-1} \\ &= (q^{\mu_r} + q^{\mu_r - 1}) - 4q^{\mu_r - 1} + q^{\mu_r-1} + q^{\mu_r-2} \\ &= q^{\mu_r-2}(q-1)^2,\end{aligned}$$

so

$$\begin{aligned}&\left|\frac{k-1}{12}A\mathcal{M}(q) - B\mathcal{P}(q)\right| \\ &\quad = \left|\frac{k-1}{12}Aq^{\nu+r-2}\left(1+\frac{1}{q}\right)(q-1)^2 - B(q^{\mu_r-2}(q-1)^2)\right| \\ &\quad = q^{\mu_r-2}(q-1)^2\left|\frac{k-1}{12}Aq^{\mu_r}\left(1+\frac{1}{q}\right) - B\right| \to \infty \quad \text{as } q \to \infty.\end{aligned}$$

When $\nu + r$ is odd, $\mu_r = \mu_{r-1} = \mu_{r-2} + 1$, so

$$\mathcal{P}(q) = 2q^{\mu_r} - 2(q^{\mu_{r-1}} + q^{\mu_{r-1}-1}) + 2q^{\mu_{r-2}} = 0,$$

and

$$\begin{aligned}&\left|\frac{k-1}{12}A\mathcal{M}(q) - B\mathcal{P}(q)\right| \\ &\quad = \left|\frac{k-1}{12}Aq^{\nu+r-2}\left(1+\frac{1}{q}\right)(q-1)^2\right| \to \infty \quad \text{as } q \to \infty.\end{aligned}$$

Case 3. $\boldsymbol{\mu_{r-2} < \nu \le \mu_r}$**.**

If $\nu + r$ is even, the condition translates to $\frac{\nu+r}{2} - 1 < \nu \le \frac{\nu+r}{2}$, so that $\mu_r = \frac{\nu+r}{2} = \nu = r$, and $\mu_{r-1} = \mu_{r-2} = \mu_r - 1$. If $\nu + r$ is odd, it translates to $\frac{\nu+r-1}{2} - 1 < \nu \le \frac{\nu+r-1}{2}$, so that $\mu_r = \mu_{r-1} = \frac{\nu+r-1}{2} = \nu = r-1$, and $\mu_{r-2} = \mu_r - 1$.

If $\nu + r$ is even, we have

$$\begin{aligned}\mathcal{P}(q) &= (q^{\mu_r} + q^{\mu_r-1}) - 4q^{r-1} + 2q^{r-2} \\ &= (q^{\nu} + q^{\nu-1}) - 4q^{\nu-1} + 2q^{\nu-2} \\ &= q^{\nu-2}(q^2 - 3q + 2) = q^{\nu-2}(q-1)(q-2),\end{aligned}$$

so

$$\begin{aligned}&\left|\frac{k-1}{12}A\mathcal{M}(q) - B\mathcal{P}(q)\right| \\ &\quad = \left|\frac{k-1}{12}Aq^{\nu+r-2}\left(1+\frac{1}{q}\right)(q-1)^2 - B(q^{\nu-2}(q-1)(q-2))\right| \\ &\quad = q^{\nu-2}(q-1)\left|\frac{k-1}{12}Aq^{r}(q-1)\left(1+\frac{1}{q}\right) - B(q-2)\right| \to \infty \quad \text{as } q \to \infty.\end{aligned}$$

If $\nu + r$ is odd, we have

$$\begin{aligned}\mathcal{P}(q) &= 2q^{\mu_r} - 2(q^{\mu_{r-1}} + q^{\mu_{r-2}}) + 2q^{r-2} \\ &= 2q^{\nu} - 2(q^{\nu} + q^{\nu-1}) + 2q^{\nu-1} = 0,\end{aligned}$$

so

$$\begin{aligned}&\left|\frac{k-1}{12}A\mathcal{M}(q) - B\mathcal{P}(q)\right| \\ &\quad = \left|\frac{k-1}{12}Aq^{\nu+r-2}\left(1+\frac{1}{q}\right)(q-1)^2\right| \to \infty \quad \text{as } q \to \infty. \qquad \square\end{aligned}$$

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