INTRODUCTION TO PROOFS
Week 1 Tutorial Solutions

Note: comments in square brackets [such as this comment] are not necessary for a complete solution.

1. Suppose X is a set.
   (a) Is it always valid to write "Take $a \in X$"?
   (b) Is it always valid to write "Suppose $a \in X$"?
   (c) What is the difference between the meanings of the words "take" and "suppose"?

   Solutions:

   (a) It is not valid to write "take $a \in X$" in the case that X is empty.

   (b) We can always suppose there is some $a \in X$, as we are not asserting there is such an a.

   (c) When we say to take something, this only makes sense when there is something to take. When we say "suppose $a \in X$", we allow ourselves to imagine what might happen should there be something in X.

2. Given a subset X of $\mathbb{Z}$, we say that a number $y \in \mathbb{Z}$ is an *upper bound for* X if, for all $x \in X$, we have $x \leq y$.

   Set
   $$X = \{x \in \mathbb{Z} : \ x < 7\}.$$
   (a) Is 10 an upper bound for X?
   (b) Is 7 an upper bound for X?
   (c) Is 7 the least, or the smallest, upper bound for X? If not, is there a least upper bound for X?
   (d) Does X have a maximal, or largest, element?
   (e) Does X have a lower bound in $\mathbb{Z}$, meaning a number $z \in \mathbb{Z}$ so that for all $x \in X$ we have $z \leq x$?

   Solution: [Note that we have not asked for explanations here.]

   (a) [For every number x in the set X, we have $x < 7$; as $7 < 10$, we have $x < 10$ and hence $x \leq 10$.] We have that 10 is an upper bound for X. [Recall that $x \leq 10$ means that either $x < 10$ or $x = 10$.]

   (b) [For every number $x \in X$, we have $x < 7$, and so $x \leq 7$.] We have that 7 is an upper bound for X.

   (c) We see that [as there are no integers between 6 and 7] for any $x \in X$, we have $x \leq 6$. So 7 is not a least upper bound for X, and in fact 6 is the least upper bound for X.

   (d) The number 6 is the maximal/largest element of X.

   (e) The set X does not have a lower bound. [Recall that $-\infty$ is not an integer.]

3. Given a subset A of $\mathbb{R}$, we say that a number $y \in \mathbb{R}$ is an *upper bound for* A if, for all $x \in A$, we have $x \leq y$.

   Set
   $$A = \{x \in \mathbb{R} : \ -1 < x < 5/2\}.$$

(a) Is 100 an upper bound for A?
(b) Now we want to show that $5/2$ is a least, or smallest, upper bound for A.
   (i) First, explain why $5/2$ is an upper bound for A.
   (ii) Now suppose that $y \in \mathbb{R}$ with $y < 5/2$. Explain why y is **not** an upper bound for A (so you need to explain why there is some element $b \in A$ with $y < b$).
(c) Does A have a greatest, or largest, lower bound? Explain your answer.

Solutions:

(a) Yes, 100 is an upper bound for A.

(b)(i) By definition, every element x of A satisfies $x < 5/2$, and so $x \leq 5/2$. Hence $5/2$ is an upper bound for A.

(b)(ii) We have assumed $y \in A$, so $y < 5/2$. Take b to be the real number half-way between y and $5/2$ [so $b = (y + 5/2)/2$]. So then $b \in \mathbb{R}$ with $b < 5/2$, meaning $b \in A$. However, $y < b$ and so y is not an upper bound for A.

(c) The number -1 is a greatest lower bound for A [although $-1 \notin A$]. To see this, suppose that $c \in \mathbb{R}$ with $c > -1$. Then we can show there are real numbers in $d \in A$ with $-1 < d < c$, and so c is not a lower bound for A. To argue this rigourously, we consider two cases. If $c \geq 5/2$ then c is larger than every number in A, and so c is not a lower bound for A. If $c < 5/2$ then with d the number half-way between -1 and c, we have $-1 < d < c \leq 5/2$, so $d \in A$ but c is not a lower bound for d (and hence c is not a lower bound for A).

4. Recall that $\mathbb{Z}_+$ is the set of *positive* integers. Discuss why any nonempty subset B of $\mathbb{Z}_+$ has a minimal, or smallest, element.

Examples: We could take B to be $\{5, 7, 8, 11, 15, 17\}$. Then 5 is the minimal element of B.

We could take B to be infinite, such as B consists of all odd positive integers larger than 16; then 17 is the minimal element of B.

[There are infinitely many examples one can consider!]