INTRODUCTION TO PROOFS
Week 3 Tutorial Solutions

Note: comments in square brackets [such as this comment] are not necessary for a complete solution.

Recall: When we write $f : X \to Y$, we are indicating that f is a function from X to Y where X, Y are necessarily nonempty sets.
Also, a function $f : X \to Y$ is injective if, for any $x, x' \in X$ with $x \neq x'$, we have $f(x) \neq f(x')$.

1. Suppose $f : X \to Y$ is injective, and fix $y \in f(X)$.
   (a) Briefly explain why there exists some $x \in X$ so that $f(x) = y$.
   (b) Now suppose $x, x' \in X$ with $f(x) = y$ and $x \neq x'$. Briefly explain why $f(x') \neq y$. (Suggestion: begin by recalling the definition of injective.)
   (c) Using (a) and (b), briefly explain why there is a **unique** $x \in X$ so that $f(x) = y$. (Recall that f is injective and $y \in f(X)$.)

   Solution:

   (a) By the definition of $f(X)$ [being $f(X) = \{f(x) : \ x \in X \}$], there must be some $x \in X$ with $f(x) = y$.

   (b) We have assumed that f is injective and $x \neq x'$. Thus $f(x) \neq f(x')$, and since $f(x) = y$, we get $y \neq f(x')$.

   (c) In (a) we showed there is some $x \in X$ with $f(x) = y$, and in (b) we showed that for any other $x' \in X$, we have $f(x') \neq y$. Thus there is exactly one $x \in X$ with $f(x) = y$.

2. Suppose $f : X \to Y$, $g : Y \to Z$, $h : Z \to W$.
   (a) Suppose $s : X \to Z$. For $x \in X$, what is $h \circ s(x)$? (Your answer should be presented in a complete sentence, so it could start "We have $h \circ s(x) =$".)
   (b) Now suppose $s = g \circ f$ and $x \in X$. According to your answer in (a), what is $h \circ (g \circ f)(x)$? Now use the definition of $g \circ f$ to further evaluate $h \circ (g \circ f)(x)$.
   (c) Supppose $t : Y \to W$. For $x \in X$, what is $t \circ f(x)$?
   (d) Now suppose $t = h \circ g$, and $x \in X$. According to your answer in (c), what is $(h \circ g) \circ f(x)$? Now use the definition of $h \circ g$ to further evaluate $(h \circ g) \circ f(x)$.
   (e) What can you conclude from (b) and (c)?

   Solution:

   (a) We have $h \circ s(x) = h(s(x))$.

   (b) We have $h \circ (g \circ f)(x) = h(g \circ f(x))$, and since $g \circ f(x) = g(f(x))$, we have $h \circ (g \circ f)(x) = h(g(f(x)))$.

   (c) We have $t \circ f(x) = t(f(x))$.

   (d) We have $(h \circ g) \circ f(x) = (h \circ g)(f(x))$ and [setting $y = f(x)$ we have $h \circ g(y) = h(g(y))$ and so] we have $(h \circ g)(f(x)) = h(g(f(x)))$.

   (e) Using (b) and (d), we see that for any $x \in X$, we have $h \circ (g \circ f)(x) = (h \circ g) \circ f(x)$. [Hence we can conclude from Theorem 1.3 that $h \circ (g \circ f) = (h \circ g) \circ f$.]

3. Negate the following statements, concluding with statements that do not use the symbol $\neg$.
   (a) $\exists m \in \mathbb{Z}$ such that $|a_m| \geq 5$.
   (b) $\forall n \in \mathbb{Z}_+, n > m \implies |a_n| < 5$.
   (c) $(\exists m \in \mathbb{Z}$ such that $|a_m| \geq 5) \wedge (\forall n \in \mathbb{Z}_+, n > m \implies |a_n| < 5)$.

Solution: [Following how the lecture notes present negating a statement:]

(a) The negation of

$$\exists m \in \mathbb{Z} \text{ such that } |a_m| \geq 5$$

is

$$\forall m \in \mathbb{Z}, \text{ we have } \neg(|a_m| \geq 5),$$

or equivalently,

$$\forall m \in \mathbb{Z}, \text{ we have } |a_m| < 5.$$

Alternatively, we can present this as follows. We have

$$\begin{aligned} &[\neg(\exists m \in \mathbb{Z} \text{ such that } |a_m| \geq 5)] \\ \iff &[\forall m \in \mathbb{Z}, \text{ we have } \neg(|a_m| \geq 5)] \\ \iff &[\forall m \in \mathbb{Z}, \text{ we have } |a_m| < 5]. \end{aligned}$$

(Note that the uses of brackets are important to make clear what statements are being claimed as equivalent.)

(b) The negation of

$$\forall n \in \mathbb{Z}_+, n > m \implies |a_n| < 5$$

is

$$\exists n \in \mathbb{Z}_+, \neg(n > m \implies |a_n| < 5)$$

or equivalently,

$$\exists n \in \mathbb{Z}_+ \text{ so that } (n > m) \wedge (|a_n| \geq 5).$$

(Recall that $\neg(P \implies Q)$ is equivalent to $P \wedge (\neg Q)$, and $\neg(|a_n| < 5)$ is $|a_n| \geq 5$.)

Alternatively, we can present this as follows.

$$\begin{aligned} &[\neg(\forall n \in \mathbb{Z}_+, n > m \implies |a_n| < 5)] \\ \iff &[\exists n \in \mathbb{Z}_+, \neg(n > m \implies |a_n| < 5)] \\ \iff &[\exists n \in \mathbb{Z}_+ \text{ so that } (n > m) \wedge (|a_n| \geq 5)]. \end{aligned}$$

(c) We have

$$\begin{aligned} &\neg[(\exists m \in \mathbb{Z} \text{ such that } |a_m| \geq 5) \wedge (\forall n \in \mathbb{Z}_+, n > m \implies |a_n| < 5)] \\ \iff &[\neg(\exists m \in \mathbb{Z} \text{ such that } |a_m| \geq 5) \vee \neg(\forall n \in \mathbb{Z}_+, n > m \implies |a_n| < 5)] \\ \iff &[(\forall m \in \mathbb{Z}, \text{ we have } |a_m| < 5) \vee (\exists n \in \mathbb{Z}_+ \text{ so that } (n > m) \wedge (|a_n| \geq 5)].) \end{aligned}$$

[Note that the phrases ”such that” and ”we have” are not strictly necessary, and are inserted to help us understand the meaning of the statements written symbolically.]

4. Let P and Q represent propositions (so P and Q each represent a statement that can either be true or false, but not both at once). Using a truth table, show that

$$[P \vee Q] \iff [\neg P \implies Q].$$

Solution: We have the following truth table.

P	Q	$(P \vee Q)$	$\neg P$	$(\neg P \implies Q)$
T	T	T	F	T
T	F	T	F	T
F	T	T	T	T
F	F	F	T	F

Thus for any combination of truth values for P and Q, we have the same truth values for $(P \vee Q)$ and for $(\neg P \implies Q)$. Hence the statements $(P \vee Q)$ and $(\neg P \implies Q)$ are equivalent.