INTRODUCTION TO PROOFS

Week 5 Tutorial Questions

Present all your answers in complete sentences, and state your assumptions.

1. Let X be a set, and $A, B, C \subseteq X$. In this exercise, you show that $B \cap C \subseteq (A \cap B) \cup (C \setminus A)$. To do this, you begin by supposing that $x \in B \cap C$.
   (a) Suppose $x \in A$. Explain why $x \in (A \cap B) \cup (C \setminus A)$. (Suggestion: begin by unwinding the assumption that $x \in B \cap C$.)
   (b) Suppose $x \notin A$. Explain why $x \in (A \cap B) \cup (C \setminus A)$.

2. Here you look at finding the inverse image of a set. Recall that with $f : X \to Y$ and $V \subseteq Y$, we have defined
   $$f^{-1}(V) = \{x \in X : \ f(x) \in V \ \}.$$
   Also recall that this does **not** mean that f has an inverse. So you need to work with the definition $f^{-1}(V)$ of the inverse image of the set V.
   (a) Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$. Let $V = \{1, 2, 3\}$. Find $f^{-1}(V)$.
   (b) Define $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ by $f((m, n)) = (m, m + n)$. Let $V = \{(k, k + 1) : \ k \in \mathbb{Z} \ \}$. Find $f^{-1}(V)$.

3. Here you consider the set $X = \{a, b, c, d\}$.
   (a) Explain why $\{\{a, b\}, \ \{b, c\}, \ \{c, d\}\}$ is **not** a partition of X.
   (b) Explain why $\{a\}, \ \{b\}, \ \{c\}, \ \{d\}$ is **not** a partition of X.
   (c) Explain why $\{\{a\}, \ \{c, d\}\}$ is **not** a partition of X.

4. Suppose $f : X \to Y$, and suppose that $\forall\, y \in f(X)$, $\exists$ a unique $x \in X$ so that $f(x) = y$. Here you show that this means that f is injective (meaning that whenever $x_1, x_2 \in X$ with $f(x_1) = f(x_2)$, we must have $x_1 = x_2$). To begin, suppose $y \in Y$.
   (a) Why is there (at least) one $x_1 \in X$ so that $f(x_1) = y$? (Suggestion: recall the definition of $f(X)$.)
   (b) Suppose $x_1, x_2 \in X$ with $f(x_1) = y$ and $f(x_2) = y$. Using the given assumptions on f, explain why $x_1 = x_2$.
   (c) Does this argument hold for any choice of $y \in Y$? Does this show that f is injective? Briefly explain your reasoning.