INTRODUCTION TO PROOFS
Week 5 Tutorial Questions

Present all your answers in complete sentences, and state your assumptions.

1. Let X be a set, and $A, B, C \subseteq X$. In this exercise, you show that $B \cap C \subseteq (A \cap B) \cup (C \setminus A)$. To do this, you begin by supposing that $x \in B \cap C$.
   (a) Suppose $x \in A$. Explain why $x \in (A \cap B) \cup (C \setminus A)$. (Suggestion: begin by unwinding the assumption that $x \in B \cap C$.)
   (b) Suppose $x \notin A$. Explain why $x \in (A \cap B) \cup (C \setminus A)$.

   Solutions:
   (a) Suppose $x \in B \cap C$. So we have $x \in B$ and $x \in C$. We are also supposing that $x \in A$; since we have $x \in B$, we have $x \in A \cap B$. As $A \cap B \subseteq (A \cap B) \cup (C \setminus A)$, this gives us $x \in (A \cap B) \cup (C \setminus A)$.
   (b) We are supposing that $x \in B \cap C$ and $x \notin A$. Thus $x \in C$ and $x \notin A$, or in other words, $x \in C \setminus A$. As $C \setminus A \subseteq (A \cap B) \cup (C \setminus A)$, this gives us $x \in (A \cap B) \cup (C \setminus A)$.

2. Here you look at finding the inverse image of a set. Recall that with $f : X \to Y$ and $V \subseteq Y$, we have defined
   $$f^{-1}(V) = \{x \in X : \ f(x) \in V \ \}.$$
   Also recall that this does **not** mean that f has an inverse. So you need to work with the definition $f^{-1}(V)$ of the inverse image of the set V.
   (a) Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$. Let $V = \{1, 2, 3\}$. Find $f^{-1}(V)$.
   (b) Define $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ by $f((m, n)) = (m, m + n)$. Let $V = \{(k, k + 1) : \ k \in \mathbb{Z} \ \}$. Find $f^{-1}(V)$.

   Solutions:
   (a) Here $f^{-1}(V) = \{1, -1, \sqrt{2}, -\sqrt{2}, \sqrt{3}, -\sqrt{3}\}$.
   (b) Here we have
$$\begin{aligned} f^{-1}(V) &= \{(m, n) \in \mathbb{Z} \times \mathbb{Z} : \ f((m, n)) \in V \ \} \\ &= \{(m, n) \in \mathbb{Z} \times \mathbb{Z} : \ (m, m + n) \in V \ \} \\ &= \{(m, n) \in \mathbb{Z} \times \mathbb{Z} : \ \exists k \in \mathbb{Z} \text{ so that } (m, m + n) = (k, k + 1) \ \} \\ &= \{(m, n) \in \mathbb{Z} \times \mathbb{Z} : \ n = 1 \ \} \\ &= \{(m, 1) : \ m \in \mathbb{Z} \ \}. \end{aligned}$$

3. Here you consider the set $X = \{a, b, c, d\}$.
   (a) Explain why $\{\{a, b\}, \ \{b, c\}, \ \{c, d\}\}$ is **not** a partition of X.
   (b) Explain why $\{a\}, \ \{b\}, \ \{c\}, \ \{d\}$ is **not** a partition of X.
   (c) Explain why $\{\{a\}, \ \{c, d\}\}$ is **not** a partition of X.

   Solutions:
   (a) A partition of X is a collection (i.e. a set) of subsets of X so that each element of X is in exactly one set in this collection. But

in the collection given, there are two sets containing b (and two sets containing c).

(b) What is given is a list of subsets of X, not a set of subsets of X.

(c) In this collection (i.e. set) of subsets of X, there is no set containing b.

4. Suppose $f : X \to Y$, and suppose that $\forall\, y \in f(X)$, $\exists$ a unique $x \in X$ so that $f(x) = y$. Here you show that this means that f is injective (meaning that whenever $x_1, x_2 \in X$ with $f(x_1) = f(x_2)$, we must have $x_1 = x_2$). To begin, suppose $y \in Y$.
   (a) Why is there (at least) one $x_1 \in X$ so that $f(x_1) = y$? (Suggestion: recall the definition of $f(X)$.)
   (b) Suppose $x_1, x_2 \in X$ with $f(x_1) = y$ and $f(x_2) = y$. Using the given assumptions on f, explain why $x_1 = x_2$.
   (c) Does this argument hold for any choice of $y \in Y$? Does this show that f is injective? Briefly explain your reasoning.

Solutions:

(a) Suppose $y \in f(X)$. So by the definition of $f(X)$, there is some $x_1 \in X$ so that $f(x_1) = y$.

(b) Suppose $x_1, x_2 \in X$ with $f(x_1) = y = f(x_2)$. By assumption, there is a unique element of X that f maps to y, so we must have $x_1 = x_2$.

(c) The above argument holds for any $y \in f(X)$. Suppose that $x_1, x_2 \in X$ with $f(x_1) = f(x_2)$. Take $y = f(x_1)$. Then by the above argument, $x_1 = x_2$, which means that f is injective.