INTRODUCTION TO PROOFS
Week 7 Tutorial Solutions

Note: comments in square brackets [such as this comment] are not necessary for a complete solution.

1. Assume X is a nonempty set with an equivalence relation $\sim$. For $x, y \in X$, let $[x]$ denote the equivalence class of x, and $[y]$ the equivalence class of y. Also suppose $z \in [x] \cap [y]$, and $w \in [y]$. Arrange the following phrases to produce a proof that $w \in [x]$. You may use phrases more than once (or not at all).
   (1) because $\sim$ is reflexive.
   (2) because $\sim$ is symmetric.
   (3) because $\sim$ is transitive.
   (4) by definition.
   (5) by assumption.
   (6) We have $y \sim z$
   (7) We have $w \sim x$
   (8) We have $w \sim y$
   (9) We have $w \sim z$
   (10) We have $x \sim z$
   (11) We have $z \sim x$ and $z \sim y$
   (12) We have $w \sim x$
   (13) We have $w \in [x]$

   Solution: One solution is to arrange the statements as: (11), (5), (6), (2), (8), (5), (9), (3), (12), (3), (13), (4)
   to give us
   We have $z \sim x$ and $z \sim y$ by assumption. We have $y \sim z$ because $\sim$ is symmetric. We have $w \sim y$ by assumption. We have $w \sim z$ because $\sim$ is transitive. We have $w \sim x$ because $\sim$ is transitive. We have $w \in [x]$ by definition.
   (Alternatively, one could use "by definition" in place of "by assumption", and one could switch the order of sentences 2 and 3.)

2. Suppose X is a set.
   (a) Suppose $A, D \subseteq X$. Show that $(A \cap D)^c = A^c \cup D^c$.
   (b) Suppose $\{B_i : i \in \mathbb{Z}_+\}$ is a collection (i.e. a set) of subsets of X. For $n \in \mathbb{Z}_+$, let $P(n)$ be the statement

   $$(B_1 \cap B_2 \cap \cdots \cap B_n)^c = B_1^c \cup B_2^c \cup \cdots \cup B_n^c.$$

   Use induction to prove that $P(n)$ holds for all $n \in \mathbb{Z}$ with $n \geq 2$. (Suggestion: for the induction step, make choices for A and D and use part (a).)

   Solution:

(a) Suppose $x \in X$. Then:

$$\begin{aligned} x \in (A \cap D)^c &\iff \neg(x \in A \cap D) \\ &\iff \neg(x \in A \wedge x \in D) \\ &\iff (x \notin A \vee x \notin D) \\ &\iff (x \in A^c \vee x \in D^c) \\ &\iff x \in A^c \cup D^c. \end{aligned}$$

Thus $(A \cap D)^c = A^c \cup D^c$.

(b) [Base case:] Taking $A = B_1$ and $D = B_2$, part (a) shows that $(B_1 \cap B_2)^c = B_1^c \cup B_2^c$. This shows that $P(2)$ holds.

[Induction step:] Now suppose $k \in \mathbb{Z}$ with $k \geq 2$, and suppose that $P(k)$ holds. Set $A = B_1 \cap B_2 \cap \cdots \cap B_k$ and $D = B_{k+1}$. Then by part (a), we have $(A \cap D)^c = A^c \cup D^c$, and so [substituting for A and D] we have

$$(B_1 \cap B_2 \cap \cdots \cap B_{\cap} B_{k+1})^c = (B_1 \cap B_2 \cap \cdots \cap B_k)^c \cup B_{k+1}^c.$$

Then using the hypothesis that $P(k)$ holds, we get

$$\begin{aligned} &(B_1 \cap B_2 \cap \cdots \cap B_k \cap B_{k+1})^c \\ &\quad = (B_1 \cap B_2 \cap \cdots \cap B_k)^c \cup B_{k+1}^c \\ &\quad = B_1^c \cup B_2^c \cup \cdots \cup B_k^c \cup B_{k+1}^c, \end{aligned}$$

showing that $P(k+1)$ holds [having assumed that $P(k)$ holds where $k \geq 2$]. Thus by the principle of mathematical induction, $P(n)$ holds for all $n \in \mathbb{Z}$ with $n \geq 2$.

[**Note:** In the above proof, one could have used n instead of k.]