INTRODUCTION TO PROOFS
Week 9 Tutorial Solutions

Note: comments in square brackets [such as this comment] are not necessary for a complete solution.

Working through this tutorial sheet, for some questions you may want to use the following.

Fact 1. By definition, a (nonempty) set X is countable if there is a bijective function $f : \mathbb{Z}_+ \to X$.

Fact 2. Corollary 8.4 says the following. Suppose $X \subseteq \mathbb{Z}_+$. Then X is finite or countable.

1. Suppose X is a set.
   (a) Suppose $A, D \subseteq X$. Show that $(A \cap D)^c = A^c \cup D^c$.
   (b) Suppose $\{B_i : \ i \in \mathbb{Z}_+ \}$ is a collection (i.e. a set) of subsets of X. For $n \in \mathbb{Z}_+$, let $P(n)$ be the statement

$$(B_1 \cap B_2 \cap \cdots \cap B_n)^c = B_1^c \cup B_2^c \cup \cdots \cup B_n^c.$$

   Use induction to prove that $P(n)$ holds for all $n \in \mathbb{Z}$ with $n \geq 2$. (Suggestion: for the induction step, make choices for A and D and use part (a).)

Solutions:

(a) Suppose $x \in X$. Then

$$\begin{aligned} x \in (A \cap D)^c &\iff \neg(x \in A \cap D) \\ &\iff \neg(x \in A \ \wedge \ x \in D) \\ &\iff (x \notin A \ \vee \ x \notin D) \\ &\iff (x \in A^c \ \vee \ x \in D^c) \\ &\iff x \in A^c \cup D^c. \end{aligned}$$

Thus the elements of X that are in $(A \cap D)^c$ are exactly the elements of X that are in $A^c \cup D^c$. Hence $(A \cap D)^c = A^c \cup D^c$.

(b) [Base case:] Suppose $B_1, B_2 \subseteq X$. Then taking $A = B_1$ and $D = B_2$ in part (a), we get $(B_1 \cap B_2)^c = B_1^c \cup B_2^c$.

[Induction step:] Now suppose that $k \in \mathbb{Z}$ with $n \geq 2$, and $B_1, \ldots, B_k, B_{k+1} \subseteq X$. Further suppose that $(B_1 \cap B_2 \cap \cdots \cap B_k)^c = B_1^c \cup B_2^c \cup \cdots \cup B_k^c$ [this is the induction hypothesis]. Setting $A = B_1 \cap B_2 \cap \cdots \cap B_k$ and setting $D = B_{k+1}$, by part (a) we have

$$(B_1 \cap \cdots \cap B_k \cap B_{k+1})^c = (B_1 \cap \cdots \cap B_k)^c \cup B_{k+1}^c.$$

Then by the induction hypothesis, we get

$$(B_1 \cap \cdots \cap B_k \cap B_{k+1})^c = B_1^c \cup \cdots \cup B_k^c \cup B_{k+1}^c.$$

Thus by the Principle of Mathematical Induction, for any $n \in \mathbb{Z}$ with $n \geq 2$, and $B_1, \ldots, B_n \subseteq X$, we have $(B_1 \cap \cdots \cap B_n)^c = B_1^c \cup \cdots \cup B_n^c$.

2. Suppose X is a countable set. Suppose A is a subset of X; prove that A is finite or countable. (Suggestion: If A is finite then we are done. So suppose A is infinite. Recall that since X is countable, there is a bijective map $f : X \to \mathbb{Z}_+$. Construct an injective map from A into $\mathbb{Z}_+$, and **prove** this map is injective.)

   Solution: [This uses the result that for an infinite set X, X is countable if and only if there exists an injective map from X into $\mathbb{Z}_+$.]

   If A is finite then we are done. So suppose A is infinite. Since X is countable, we know there is a bijective map $f : X \to \mathbb{Z}_+$. Let $h : A \to X$ be defined by $h(a) = a$; so [as we have seen before] h is injective. [To prove h is injective: Suppose $a, a' \in A$ with $a \neq a'$. Then $h(a) = a \neq a' = h(a')$, and hence h is injective.] Thus the map $f \circ h : A \to \mathbb{Z}_+$ is injective [since h, f are injective], and since A is infinite, this means A must be countable.

3. Let $A = \{x \in \mathbb{Z}_+ : \ x \text{ is even } \}$ and let $B = \{x \in \mathbb{Z}_+ : \ x \text{ is odd } \}$. We know that A and B are infinite subsets of $\mathbb{Z}_+$, and $\mathbb{Z}_+$ is countable; so by Corollary 8.4, A and B are countable. Now suppose that X is a countable set. Show that there is a subset C of X so that C and $X \setminus C$ are both countable. (Recall that a countable set is necessarily infinite. Suggestion: begin with the definition of X being countable, then use this to identify A with a countable subset of X.)

   Solution: Since X is countable, there is a bijection $f : \mathbb{Z}_+ \to X$. Let $C = f(A)$. Since f is bijective, it gives us a bijection between A and C, so $|A| = |C|$. As A is countable, this means C is also countable.

   We know that $B = \mathbb{Z}_+ \setminus A$. We claim that $f(B) \subseteq X \setminus C$. Suppose $y \in f(B)$. Thus for some $x \in B$, we have $y = f(x)$. For the sake of contradiction, suppose $y \notin X \setminus C$; since $y \in f(B) \subseteq f(\mathbb{Z}_+) = X$, we must have $y \in C$. As $C = f(A)$ and $y \in C$, we must have $y = f(z)$ for some $z \in A$. Thus we have $f(x) = y = f(z)$; since f is bijective and thus injective, this means that $x = z$. So $x \in B$ and $x = z \in A$. However, $A \cap B = \emptyset$, so no such x can exist. Thus $y \in X \setminus C$. This argument holds for all $y \in f(B)$, and hence $f(B) \subseteq X \setminus C$.

   Since B is countable and f is bijective, we have that $|f(B)|$ is countable. Also, as $f(B) \subseteq X \setminus C \subseteq X$ and X is countable, we have

$$|\mathbb{Z}_+| = |f(B)| \leq |X \setminus C| \leq |X| = |\mathbb{Z}_+|.$$

   So $|X \setminus C| = |||_+|$, meaning $X \setminus C$ is countable.