SUMS OF SQUARES OVER FUNCTION FIELDS

Kathy Merrill and Lynne H. Walling

Given a polynomial α with coefficients in a finite field $\mathbb{F}$, how many ways can we represent α as a sum of k squares? The answer to this question is all too often "infinity." Thus instead we ask: What is the value of the "restricted representation number"

$$r(\alpha, m) = \#\left\{ (\beta_1, \dots, \beta_k) : \sum_j \beta_j^2 = \alpha \text{ and } \deg \beta_j < m \right\}?$$

Eisenstein [ref?] approached the analogous problem over $\mathbb{Z}$ using the arithmetic theory of quadratic forms; this approach was developed further by Smith and Minkowski [ref?], who were able to present formulas to solve the problem for $k \le 8$. Alternatively, Jacobi used the theory of elliptic functions and even powers of the classical theta series, $\theta(z)$, solving the problem for $k = 2, 4, 6, 8$. Hardy called Jacobi's approach "simpler" than that of Smith and Minkowski, and remarked that "it has another very important merit, that it can be used – within the limits of human capacity for calculation – for *any* even value of s" (here $s = k$) [ref: paper starts on p. 340]. Then in [ref – same as above], Hardy used the theory of elliptic functions to treat the case where k =5 or 7. Hardy wrote that the solution to the problem for $k > 8$ involves "other and more recondite arithmetical functions" [ref, p. 340?]; still, in [ref #4 cited on p. 344], Hardy and Ramanujan introduced techniques that led to asymptotic formulas for these representation numbers with k arbitrary ($k > 4$).

We note here that in light of our present knowledge of holomorphic automorphic forms, it is not surprising that Hardy and Ramanujan obtain only asymptotic formulas for $k > 8$. Their approach involves the study of $\theta(z)^k$, which is a holomorphic automorphic form of weight $\frac{k}{2}$ for the congruence subgroup $\Gamma_0(4)$. When $k \le 8$ and k is even, we know there are no cusp forms for $\Gamma_0(4)$ and thus $\theta(z)^k$ must be an Eisenstein series (whose Fourier coefficients are well understood and easily computed). However, when $k > 8$, $\theta(z)^k$ is a linear combination of an Eisenstein series and a cusp form – Hardy's "recondite" function. The order of magnitude of the Fourier coefficients of a cusp form is small compared to that of an Eisenstein series; thus the Fourier coefficients of $\theta(z)^k$ are asymptotic to those of the associated Eisenstein series. (To find the dimension of a space of integral weight cusp forms with level and character, one can use the formula found in [Ross] which is derived from Hijikata's trace formula.)

---

second author partially supported by NSF grant DMS 9103303

In this paper we too will use powers of a theta function to study the restricted representation numbers $r(\alpha, m)$ where α lies in the polynomial ring $\mathbb{F}[T]$ (T an indeterminate); the theta function $\theta(z)$ we use was recently presented in [H-R] (see Thm?). After some preliminary remarks, we show that $\theta(z)$ transforms like an automorphic form of weight 1/2 under the "full modular group" Γ (see Theorem 2.4). Then using rather elementary techniques, we derive a formula for $r(\alpha, m)$. This formula involves Kloosterman sums when $\deg \alpha \geq 4$, but we are able to compute: (1) the average value of $r(\alpha, m)$; (2) the order of magnitude of $r(\alpha, m)$ as $m \to \infty$ or $\deg \alpha \to \infty$; and (3) an asymptotic formula for $r(\alpha, m)$ as k, the number of squares, approaches ∞ (see Theorems 3.11, 3.14, 3.15 resp.).

For a full account of the history of this problem over $\mathbb{Z}$, the reader is refered to [Grosswald]. To read about automorphic forms over a function field, the reader is refered to [Weil] and [H-R].

The authors thank Jeff Hoffstein and Jeff Stopple for many helpful conversations.

§1. **Preliminaries.** One of the most intriguing and most studied structures in number theory is $\mathbb{Z}$, the ring of rational integers. The theory of automorphic forms has provided us with some powerful tools to aid us in our study of $\mathbb{Z}$; see, for example, [Terras]. We review here the basic classical set-up.

To each of the valuations on $\mathbb{Q}$, the field of fractions of $\mathbb{Z}$, we can associate an "upper half-plane" $\mathfrak{H} = G/K$. Here $G = PSL_2$(completion of $\mathbb{Q}$) and K is the maximal compact subgroup of G. An automorphic form on $\mathfrak{H}$ is a function which transforms with a factor of "automorphy" under the action of $\Gamma = SL_2(\mathbb{Z})$ on $\mathfrak{H}$, and is invariant under the subgroup $\Gamma_\infty = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : \ a \in \mathbb{Z} \right\}$ (see [Shimura's book] or [Koblitz]).

We want to parallel this arrangement to study the polynomial ring, $\mathbb{A} = \mathbb{F}[T]$; here $\mathbb{F}$ is a finite field and T is an indeterminate. For the sake of clarity, we treat only the case where $\mathbb{F}$ has p elements, p an odd prime. We denote the field of fractions of $\mathbb{A}$ by $\mathbb{K} = \mathbb{F}(T)$. One of the valuations $|\cdot|_\infty$ on $\mathbb{K}$, the "infinite" valuation, is induced by the degree map: for $\alpha, \beta \in \mathbb{A}$, define

$$|\alpha/\beta|_\infty = p^{\deg \alpha - \deg \beta}.$$

We adopt the convention that $\deg 0 = -\infty$, and hence $|0|_\infty = 0$. Note that unlike the infinite valuation on $\mathbb{Q}$ – which is absolute value – this infinite valuation is nonarchimedean; this in fact eases many computations. We let $\mathbb{K}_\infty$ denote the completion of $\mathbb{K}$ with respect to $|\cdot|_\infty$; one easily sees that $\mathbb{K}_\infty = \mathbb{F}((\frac{1}{T}))$, formal Laurent series in $\frac{1}{T}$. The "unit ball" or "ring of integers" in $\mathbb{K}_\infty$ is

$$\mathcal{O}_\infty = \{x \in \mathbb{K}_\infty : \ |x|_\infty \leq 1 \ \} = \mathbb{F}[[\tfrac{1}{T}]]$$

(so $\mathcal{O}_\infty$ consists of formal Taylor series in $\frac{1}{T}$). We note here that $\mathcal{O}_\infty$ is a discrete valuation ring with a unique maximal ideal $\mathfrak{P}_\infty = \{x \in \mathbb{K}_\infty : \ |x|_\infty < 1 \ \}$.

Let μ be (additive) Haar measure on $\mathbb{K}_\infty$, normalized so that $\mu(\mathcal{O}_\infty) = 1$. By the translation invariance of μ, we have that for any $a \in \mathbb{K}_\infty$,

$$\mu\left(\{x : \ x_j = a_j \text{ for } j > n \ \}\right) = p^n.$$

(Here $x = \sum_{j \geq -\infty} x_j T^j$.)

Set $G = PSL_2(\mathbb{K}_\infty)$; then the maximal compact subgroup of G (with respect to the standard? topology induced on G by $|\cdot|_\infty$) is $PSL_2(\mathcal{O}_\infty)$. Thus we set

$$\mathfrak{H} = PSL_2(\mathbb{K}_\infty)/PSL_2(\mathcal{O}_\infty).$$

In §2 we will define an automorphic form on $\mathfrak{H}$.

Remark. Some authors, like Weil and Ephrat, use PGL instead of PSL in the definition of $\mathfrak{H}$. While this choice is irrelevent in the construction of the complex upper half-plane, it is not irrelevent in the function field setting.

Proposition 1.1. *The set*

$$\left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} : \ y = T^{2m}, \ m \in \mathbb{Z}, \ x \in T^{2m+1}\mathbb{A} \ \right\}$$

is a complete set of coset representatives for $\mathfrak{H} = PSL_2(\mathbb{K}_\infty)/PSL_2(\mathcal{O}_\infty)$.

Proof. Take $z = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{K}_\infty)$. Then

$$z \equiv \begin{cases} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{c}{d} & 1 \end{pmatrix} & \text{if } \deg c \leq \deg d, \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \frac{d}{c} & 1 \\ -1 & 0 \end{pmatrix} & \text{if } \deg d < \deg c \end{cases}$$

where $z \equiv z'$ means z and z' represent the same coset of $\mathfrak{H}$. Thus z is equivalent to a matrix of the form

$$\begin{pmatrix} w & x' \\ 0 & w^{-1} \end{pmatrix}$$

with $w, x' \in \mathbb{K}_\infty$, $w \neq 0$. Now, $w = T^m u$ for some $m \in \mathbb{Z}$ and $u \in \mathcal{O}_\infty^\times$. So

$$z \equiv \begin{pmatrix} w & x' \\ 0 & w^{-1} \end{pmatrix} \begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix} \equiv \begin{pmatrix} T^m & x'' \\ 0 & T^{-m} \end{pmatrix}$$

for some $x'' \in \mathbb{K}_\infty$. Writing x'' as $T^{-m}(x + T^{2m}v)$ where $x \in T^{2m+1}\mathbb{A}$, $v \in \mathcal{O}_\infty$, we see that

$$z \equiv \begin{pmatrix} T^m & x'' \\ 0 & t^{-m} \end{pmatrix} \begin{pmatrix} 1 & -v \\ 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} T^m & T^{-m}x \\ 0 & T^{-m} \end{pmatrix} \equiv \begin{pmatrix} T^{2m} & x \\ 0 & 1 \end{pmatrix}.$$

Now suppose $z \equiv \begin{pmatrix} T^{2m} & x \\ 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} T^{2m'} & x' \\ 0 & 1 \end{pmatrix}$ where $m, m' \in \mathbb{Z}$, $x \in T^{2m+1}\mathbb{A}$, $x' \in T^{2m'+1}\mathbb{A}$. Then

$$\begin{pmatrix} T^m & T^{-m}x \\ 0 & T^{-m} \end{pmatrix}^{-1} \begin{pmatrix} T^{m'} & T^{-m'}x' \\ 0 & T^{-m'} \end{pmatrix} = \begin{pmatrix} T^{m'-m} & T^{1m'-m}(x'-x) \\ 0 & T^{m-m'} \end{pmatrix} \in SL_2(\mathcal{O}_\infty).$$

Thus $T^{m'-m}, T^{m-m'} \in \mathcal{O}_\infty$, which implies $m = m'$. So $T^{-m'-m}(x' - x) = T^{-2m}(x' - x) \in \mathcal{O}_\infty$. We know $x, x' \in T^{2m+1}\mathbb{A}$, so

$$T^{-2m}(x' - x) \in \mathcal{O}_\infty \cap T\mathbb{A} = \{0\}\,.$$

Hence $x = x'$. □

The group $\Gamma = SL_2(\mathbb{A})$ acts on $\mathfrak{H}$ by left multiplication. We expect certain functions, which we will call automorphic, should transform with some sort of factor of automorphy; these functions should actually be invariant under the action of

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} : \ \alpha \in \mathbb{A} \right\}.$$

For such a function f, we fix $y = T^{-2m}$ and let f_y be the resulting function on $T^{1-2m}\mathbb{A}$ given by $f_y(x) = f\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right)$. Because of the invariance of f under Γ_∞, we can consider f_y as a function on the finite abelian (additive) subgroup $T^{1-2m}\mathbb{A}/\mathbb{A}$ of $\mathbb{K}_\infty/\mathbb{A}$. This makes Fourier series in x a useful tool for analyzing automorphic forms. Before defining the specific automorphic form we will study, we give a description of Fourier series in this setting.

For $x \in \mathbb{K}_\infty$, write $x = \sum_{j=-\infty}^{n} x_j T^j$, and let $e\{x\} = exp\{2\pi i x_1/p\}$. We use this definition on $\mathbb{K}_\infty/\mathbb{A}$ as well, by making the natural identification between $\mathbb{K}_\infty/\mathbb{A}$ and $\{x \in \mathbb{K}_\infty : x_j = 0 \text{ for } j \geq 0\}$.

Lemma 1.2. *The character groups of $\mathbb{K}_\infty$, $\mathbb{K}_\infty/\mathbb{A}$, and $T^{1-2m}\mathbb{A}/\mathbb{A}$ are isomorphic to $\mathbb{K}_\infty$, $T^2\mathbb{A}$, and $T^2\mathbb{A}/T^{2m+1}\mathbb{A}$ respectively.*

Proof. For $\beta \in \mathbb{K}_\infty$, let ψ_β be defined by $\psi_\beta(x) = e\{\beta x\}$. Each ψ_β is clearly a continuous homomorphism on $\mathbb{K}_\infty$; by restricting to subgroups of monomials, we see that all characters of $\mathbb{K}_\infty$ are of this form. The characters of $\mathbb{K}_\infty/\mathbb{A}$ consist of those ψ_β which are trivial on $\mathbb{A}$, that is $\{\psi_\beta : \beta \in T^2\mathbb{A}\}$. The characters of $T^{1-2m}\mathbb{A}/\mathbb{A}$ are then obtained by equating characters of $\mathbb{K}_\infty/\mathbb{A}$ which agree on $T^{1-2m}\mathbb{A}/\mathbb{A}$.

Elementary harmonic analysis then leads to the following:

Theorem 1.3. *Any function f on $\mathfrak{H}$ which is invariant under the action of Γ_∞ can be expanded in a Fourier series $f\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = \sum_{\beta \in T^2\mathbb{A}} c_\beta(y)\chi(\beta y)e\{\beta x\}$, where $c_\beta(T^{-2m}) = p^{1-2m} \sum_{x \in T^{1-2m}\mathbb{A}/\mathbb{A}} f\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) e\{-\beta x)$ and $\chi = \chi_{\mathcal{O}_\infty}$ is the characteristic function of $\mathcal{O}_\infty$.*

Proof. We fix $y = T^{-2m}$, and use the Lemma to write

$$f\left(\begin{pmatrix} T^{-2m} & x \\ 0 & 1 \end{pmatrix}\right) = f_y(x) = \sum_{\beta \in T^2\mathbb{A}/T^{2m+1}\mathbb{A}} c_\beta(T^{-2m}) e\{\beta x\}$$

where $c_\beta(T^{-2m}) = p^{1-2m} \sum_{x \in T^{1-2m}\mathbb{A}/\mathbb{A}} f_y(x)e\{-\beta x)$, and where we make the natural identification between $\beta \in T^2\mathbb{A}/T^{2m+1}\mathbb{A}$ and $\{\beta \in T^2\mathbb{A} : \beta_j = 0 \text{ for } j > 2m\}$. The theorem then follows by noting that $\beta \in T^2\mathbb{A}$ satisfies $deg\beta \leq 2m$ if and only if $\beta y \in \mathcal{O}_\infty$.

While Fourier series of the above form are most central in providing the shape of our theta function, we will also occasionally make use of Fourier analysis on $\mathbb{K}_\infty/\mathbb{A}$ and on all of $\mathbb{K}_\infty$. Thus we remark here that techniques parallel to those used in the classical setting allow us to define the Fourier transform of a function $f \in L^1(\mathbb{K}_\infty)$

by $\hat{f}(t) = \int_{\mathbb{K}_\infty} f(x)e\{-tx\}dx$, where the inversion formula $f(x) = \int_{\mathbb{K}_\infty} \hat{f}(t)e\{tx\}dt$ (in L^1) holds for sufficiently nice functions. Similarly, for $f \in L^1(\mathbb{K}_\infty/\mathbb{A})$, we write $\hat{f}(\beta) = p\int_{\mathbb{K}_\infty/\mathbb{A}} f(x)e\{-\beta x\}dx$ and obtain $f(x) = \sum_{\beta\in T^2\mathbb{A}} \hat{f}(\beta)e\{\beta x\}$. We will apply these formulae in settings where convergence is easily established.

§**2. The theta series.** Now that we know the form of a Fourier series we can define our theta series. In analogy with the classical theta series, we form the simplest possible Fourier series attached to squares of the elements of a rank 1 $\mathbb{A}$-lattice, or the shift of such a lattice.

Fix $r \in \mathbb{Z}$ and set $L = T^r\mathbb{A}$. We define the (homogeneous) theta series attached to L to be

$$\theta(L; z) = \sum_{\ell\in L} \chi(y\ell^2)e\left\{x\ell^2\right\}.$$

For $h \in \mathbb{K}_\infty$, we define an inhomogeneous theta series:

$$\theta(L, h; z) = \sum_{\ell\in L} \chi(y(\ell + h)^2)\ e\left\{x(\ell + h)^2\right\}.$$

(Here $z = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \in \mathfrak{H}$.) By $L^\#$ we denote the dual of L:

$$L^\# = \{x \in \mathbb{K}_\infty :\ e\{x\ell\} = 1 \text{ for all } \ell \in L\ \} = T^{2-r}\mathbb{A}.$$

Note that when $L = T\mathbb{A}$, we have $L^\# = L = T\mathbb{A}$. Hence we consider the lattice $T\mathbb{A}$ to be fundamental, and we let

$$\theta(z) = \theta(T\mathbb{A}; z).$$

We will eventually show that $\theta(z)$ transforms under Γ. We first need to establish some technical Lemmas, and then we need to find an inversion formula for inhomogeneous theta series.

Lemma 2.1. *Let $x \in \mathbb{K}_\infty^\times$, and take $t \in \mathbb{Z}$. Then*

$$\int_{\mathfrak{P}_\infty^t} e\{xb\}\ db = \begin{cases} p^{-t} & \text{if } \deg x \le t \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If $\deg x \le t$ then $e\{xb\} = 1$ for any $b \in \mathfrak{P}_\infty^t$ and so the integral is equal to the measure of $\mathfrak{P}_\infty^t$. Now suppose $n = \deg x > t$. Note that $e\{xb\} = e\{xb'\}$ whenever $b - b' \in \mathfrak{P}_\infty^n$. Thus

$$\begin{aligned}\int_{\mathfrak{P}_\infty^t} e\{xb\}\ db &= \mu(\mathfrak{P}_\infty^n) \sum_{b\in\mathfrak{P}_\infty^t/\mathfrak{P}_\infty^n} e\{xb\} \\ &= p^{-n} \sum_{b_{-t},\dots,b_{1-n}\in\mathbb{F}} e\{T(x_n b_{1-n} + \cdots + x_{1+t}b_{-t})\} \\ &= 0\end{aligned}$$

since $x_n \neq 0$ and hence the sum on b_{1-n} is a nontrivial character sum. □

Lemma 2.2. *Take $v \in \mathcal{O}_\infty^\times$ and $n \in \mathbb{Z}_+$. Then*

$$\sum_{c \in \mathcal{O}_\infty / \mathfrak{P}_\infty^n} e\left\{T^n v c^2\right\} = p^{\frac{n}{2}} (v_0|p)^n \sqrt{(-1|p)^n}$$

where we take the negative real axis for our branch cut for the squareroot function.

Proof. One easily sees that

$$\sum_{c \in \mathcal{O}_\infty / \mathfrak{P}_\infty^n} e\left\{T^n v c^2\right\} = \sum_{\substack{c \in \mathcal{O}_\infty / \mathfrak{P}_\infty^{n-1} \\ d \in \mathfrak{P}_\infty^{n-1} / \mathfrak{P}_\infty^n}} e\left\{T^n v (c^2 + 2cd + d^2)\right\}.$$

When $n > 1$, $e\left\{T^n v d^2\right\} = 1$ for $d \in \mathfrak{P}_\infty^{n-1}$; also, $d \mapsto e\{2T^n cdv\}$ is a character on $\mathfrak{P}_\infty^{n-1}/\mathfrak{P}_\infty^n$ which is trivial exactly when $c \in \mathfrak{P}_\infty$. So for $n > 1$,

$$\begin{aligned} \sum_{c \in \mathcal{O}_\infty / \mathfrak{P}_\infty^n} e\left\{T^n v c^2\right\} &= p \sum_{c \in \mathfrak{P}_\infty / \mathfrak{P}_\infty^{n-1}} e\left\{T^n v c^2\right\} \\ &= p \sum_{c \in \mathcal{O}_\infty / \mathfrak{P}_\infty^{n-2}} e\left\{T^{n-2} v c^2\right\}. \end{aligned}$$

Arguing by induction on n we find that

$$\sum_{c \in \mathcal{O}_\infty / \mathfrak{P}_\infty^n} e\left\{T^n v c^2\right\} = \begin{cases} p^{\frac{n}{2}} & \text{if } n \text{ is even,} \\ p^{\frac{n-1}{2}} \sum_{c \in \mathcal{O}_\infty / \mathfrak{P}_\infty} e\left\{T v c^2\right\} & \text{otherwise.} \end{cases}$$

Now,

$$\begin{aligned} \sum_{c \in \mathcal{O}_\infty / \mathfrak{P}_\infty} e\left\{T v c^2\right\} &= \sum_{c \in \mathbb{F}} exp\left\{2\pi i v_0 c^2 / p\right\} \\ &= (v_0|p) \sqrt{(-1|p)} p^{\frac{1}{2}} \end{aligned}$$

where the last equality follows from the evaluation of the Gauss sum $\sum_{c \in \mathbb{F}} exp\left\{2\pi i v_0 c^2/p\right\}$ (see §9.9 of [Apostol]). □

With these Lemmas in hand, we merely use Poisson summation to prove the inversion formula (cf. [classical ref?]).

Theorem 2.3 (Inversion Formula). *Let $L = T^r \mathbb{A}$ with $r \geq 1$, and fix $h \in L^\#$. Let $-\frac{1}{z}$ denote $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} z$ where $z = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \in \mathfrak{H}$, $y = T^{2m}$, and $x \in T^{2m+1}\mathbb{A}$ with degree n. Then*

$$\theta(L, h; z) = p^{1-r} \frac{1}{\sqrt{z}} \sum_{s \in L^\# / L} e\{2sh\}\, \theta\left(L, s; -\frac{1}{z}\right)$$

where

$$\sqrt{z} = \begin{cases} p^{-m} & \text{if } x = 0, \\ p^{-\frac{n}{2}} (x_n|p)^n \sqrt{(-1|p)^n} & \text{if } x \neq 0. \end{cases}$$

Proof. We first establish the Theorem when $h \in L^{\#}$. Choose $z = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \in \mathfrak{H}$. We can assume $y = T^{2m}$ for some $m \in \mathbb{Z}$, and $x \in T^{2m+1}\mathbb{A}$. If $x = 0$ then $\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} y & y \\ 0 & 1 \end{pmatrix}$; thus, replacing x with y if necessary, we can assume $\deg x = n \geq 2m$. For $b \in \mathbb{K}_\infty$, let

$$\phi(b) = \chi(y(b+h)^2)\ e\left\{x(b+h)^2\right\}.$$

Then for $s \in \mathbb{K}_\infty$, the Fourier transform of ϕ is

$$\hat{\phi}(s) = \int_{\mathbb{K}_\infty} \phi(b) e\left\{-sb\right\}\ db.$$

We are using additive Haar measure, so we can replace b with $b - h$; thus

$$\begin{aligned} \hat{\phi}(s) &= e\left\{sh\right\} \int_{\mathbb{K}_\infty} \chi(T^{2m}b^2)\ e\left\{xb^2\right\}\ e\left\{-sb\right\}\ db \\ &= e\left\{sh\right\} \int_{\mathfrak{P}_\infty^m} e\left\{xb^2 - sb\right\}\ db \end{aligned}$$

where the last equality hold since

$$\chi(T^{2m}b^2) = \begin{cases} 1 & \text{if } b \in \mathfrak{P}_\infty^m, \\ 0 & \text{otherwise.} \end{cases}$$

Now, suppose $c \in \mathfrak{P}_\infty^m$ and $b \in \mathfrak{P}_\infty^{n-m}$. Then $\deg 2xbc \leq 0$ and $\deg xb^2 \leq 2m - n \leq 0$; hence

$$\begin{aligned} &\int_{\mathfrak{P}_\infty^m} e\left\{xb^2 - sb\right\}\ db \\ &\quad = \sum_{c \in \mathfrak{P}_\infty^m / \mathfrak{P}_\infty^{n-m}} e\left\{xc^2 - sc\right\} \int_{\mathfrak{P}_\infty^{n-m}} e\left\{-sb\right\}\ db \end{aligned}$$

and by Lemma 2.1,

$$= \begin{cases} p^{m-n} \displaystyle\sum_{c \in \mathfrak{P}_\infty^m / \mathfrak{P}_\infty^{n-m}} e\left\{xc^2 - sc\right\} & \text{if } \deg s \leq n - m, \\ 0 & \text{otherwise.} \end{cases}$$

So suppose $\deg s \leq n - m$. Then $\frac{s}{2x} \in \mathfrak{P}_\infty^m$, and thus

$$\begin{aligned} \sum_{c \in \mathfrak{P}_\infty^m / \mathfrak{P}_\infty^{n-m}} e\left\{xc^2 - sc\right\} &= \sum_c e\left\{x(c - \frac{s}{2x})^2\right\} e\left\{-\frac{s^2}{4x}\right\} \\ &= \sum_c e\left\{xc^2\right\} e\left\{-\frac{s^2}{4x}\right\}. \end{aligned}$$

By Lemma 2.2,

$$\sum_c e\left\{xc^2\right\} = p^{-m+\frac{n}{2}}(x_n|p)^n\sqrt{(-1|p)^n}.$$

So

$$\hat{\phi}(s) = \begin{cases} e\left\{sh - \frac{s^2}{4x}\right\} p^{-\frac{n}{2}}(x_n|p)^n\sqrt{(-1|p)^n} & \text{if } \deg s \le n-m, \\ 0 & \text{otherwise.} \end{cases}$$

Now we mimic the classical technique of Poisson summation. Define the function ξ on $\mathbb{K}_\infty/L$ by

$$\xi(t) = \sum_{b\in L}\phi(t+b).$$

We note that

$$\hat{\xi}(s) = \frac{1}{\mu(\mathbb{K}_\infty/L)}\sum_{s\in L^\#}\hat{\phi}(s)$$

for $s \in L^\#$. Thus, by the calculation above, ξ has a finite Fourier series, so that $\xi(t) = \sum_{s\in L^\#}\hat{\xi}(s)e\{st\}$ pointwise. Therefore, at $t = 0$, we obtain

$$\sum_{b\in L}\phi(b) = (vol\mathbb{K}_\infty/L)^{-1}(??)\sum_{s\in L^\#}\hat{\phi}(s)$$

and replacing s by $2s$,

$$= p^{1-r}p^{-\frac{n}{2}}(x_n|p)^n\sum_{s\in L^\#}e\left\{2sh\right\}\chi(T^{2m-2n}s^2)e\left\{-\frac{1}{x}s^2\right\}$$

$$= p^{1-r}p^{-\frac{n}{2}}(x_n|p)^n\sum_{\substack{s\in L^\#/L\\ \ell\in L}}e\left\{2sh\right\}\chi(T^{2m-2n}(s+\ell)^2)e\left\{-\frac{1}{x}(s+\ell)^2\right\}.$$

Thus

$$\theta(L,h;z) = p^{1-r}p^{-\frac{n}{2}}(x_n|p)^n\sqrt{(-1|p)^n}\sum_{s\in L/L^\#}e\left\{2sh\right\}\theta(L,s;-\frac{1}{z}).$$

Now suppose $h \notin L^\#$. Choose $a \in \mathbb{Z}_+$ such that $T^{2a}h \in L^\#$. Letting $T^{-2a}z$ denote $\begin{pmatrix} T^{-2a} & 0 \\ 0 & 1\end{pmatrix} z$, we see that $\theta(L,h;z) = \theta(T^aL, T^ah; T^{-2a}z)$. Applying the above formula to $\theta(T^aL, T^ah; T^{-2a}z)$ establishes the Theorem for all h. □

As in the classical case, it is relatively simple to derive a transformation formula once one has an inversion formula. Thus our proof of the transformation formula for function fields follows classical lines (see, for instance, Thm? [Eichler?]).

Theorem 2.4 (Transformation Formula). *Let $L = T\mathbb{A}$ and $\theta(z) = \theta(L,0;z)$. For $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma$ and $z \in \mathfrak{H}$, let $\frac{az+b}{cz+d}$ denote $\begin{pmatrix} a & b \\ c & d\end{pmatrix} z$. Let $z' = \begin{pmatrix} \frac{1}{d} & 0 \\ c & d\end{pmatrix} z$. Then*

$$\theta\left(\frac{az+b}{cz+d}\right) = \frac{\sqrt{z}}{\sqrt{z'}}\chi(d)\theta(z)$$

where

$$\chi(d) = p^{-\deg d} \sum_{\ell \in L/dL} e\left\{-\frac{c}{d}\ell^2\right\}.$$

Proof. If $d = 0$, then $c \neq 0$ and we can replace $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with

$$\begin{pmatrix} a & a+b \\ c & c \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Since $\theta(L; z)$ is invariant under the action of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, we can assume $d \neq 0$.

Note that $\frac{az+b}{cz+d} = z' + \frac{b}{d} = \begin{pmatrix} 1 & \frac{b}{d} \\ 0 & 1 \end{pmatrix} z'$ where $z' = \frac{z}{dcz+d^2} = \begin{pmatrix} \frac{a}{d} & 0 \\ c & d \end{pmatrix} z$. Thus

$$\begin{aligned} \theta(L; \frac{az+b}{cz+d}) &= \theta(L; z' + \frac{b}{d}) \\ &= \sum_{\ell \in L} \chi(y'\ell^2) e\{x'\ell^2\} e\left\{\frac{b}{d}\ell^2\right\} \\ &= \sum_{\ell_0 \in L/dL} e\left\{\frac{b}{d}\ell_0^2\right\} \ \theta(dL, \ell_0; z') \end{aligned}$$

and by the inversion formula,

$$\begin{aligned} &= p^{-\deg d} \frac{1}{\sqrt{z'}} \sum_{\substack{\ell_0 \in L/dL \\ s \in (dL)^{\#}/dL}} e\left\{\frac{b}{d}\ell_0^2 + 2s\ell_0\right\} \theta(L, \frac{s}{d}; -\frac{1}{z'}) \\ &= p^{-\deg d} \frac{1}{\sqrt{z'}} \sum_{\substack{\ell_0 \in L/dL \\ s \in L^{\#}/d^2L}} e\left\{\frac{b}{d}\ell_0^2 + 2\frac{s}{d}\ell_0\right\} \theta(d^2L, s; -\frac{1}{z} - \frac{c}{d}) \\ &= p^{-\deg d} \frac{1}{\sqrt{z'}} \sum_{\substack{\ell_0 \in L/dL \\ s \in L^{\#}/d^2L}} e\left\{\frac{b}{d}\ell_0^2 + 2\frac{s}{d}\ell_0 - \frac{c}{d}s^2\right\} \theta(d^2L, s; -\frac{1}{z}). \end{aligned}$$

Now, for $s \in L^{\#}$, $\ell_0 \in L$,

$$e\left\{\frac{b}{d}\ell_0^2 + 2\frac{s}{d}\ell_0 - \frac{c}{d}s^2\right\} = e\left\{-\frac{c}{d}(b\ell_0 + s)\right\}$$

since $-\frac{b^2c}{d}\ell_0^2 = \frac{b-ad}{d}\ell_0^2$, $a\ell_0^2 \in T^2\mathbb{A}$, $-\frac{2bc}{d}s\ell_0 = \frac{2-2ad}{d}s\ell_0$ and $as\ell_0 \in T^2\mathbb{A}$. Also, $b\ell_0 + s$ runs over $L^{\#}/dL^{\#}$ as ℓ_0 runs over L/dL (this is easily verified). Thus

$$\theta(L; \frac{az+b}{cz+d}) = p^{-\deg d} \frac{1}{\sqrt{z'}} \sum_{\ell \in L^{\#}/dL^{\#}} e\left\{-\frac{c}{d}\ell^2\right\} \ \theta(L^{\#}; 1\frac{1}{z})$$

and again by the inversion formula,

$$= p^{-\deg d} \frac{\sqrt{z}}{\sqrt{z'}} \sum_{\ell \in L/dL} e\left\{-\frac{c}{d}\ell^2\right\} \theta(L; z)$$

(recall that we have chosen L so that $L^{\#} = L$). □

Keeping in mind Shimura's work on half-integral weight automorphic forms ([ref]), we define an automorphic form to be a function $f : \mathfrak{H} \to \mathbb{C}$ which transforms like an appropriate power of $\theta(z)$ under the action of Γ. More precisely, we make the following

Definition. A function $f : \mathfrak{H} \to \mathbb{C}$ is an automorphic form of weight $k \in \frac{1}{2}\mathbb{Z}_+$ if for all $\gamma \in \Gamma$,

$$f(\gamma z) = j(\gamma, z)^{2k} f(z)$$

where $j(\gamma, z) = \frac{\theta(\gamma z)}{\theta(z)}$.

Remarks. Just as in the classical case, we can define

$$E_k(z) = \sum_{\gamma} j(\gamma, z)^{2k}$$

where γ varies over Γ_∞/Γ; the series converges absolutely for $k > 2$ and is easily seen to be an automorphic form of weight k. (Mention Poincare series? Petersson inner product?) We can also define Hecke operators T_α as follows. For $\alpha \in \mathbb{A}$ with $\deg \alpha$ even, let $\{\gamma_i\}$ be a complete set of coset representatives for $(\Gamma \cap \Gamma^\alpha)/\Gamma^\alpha$ where

$$\Gamma^\alpha = \begin{pmatrix} 1 & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \Gamma \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}.$$

Then T_α maps a wieght k automorphic form f to the weight k automorphic form given by

$$p^{-\deg \alpha} \sum_i j(\gamma_i, \begin{pmatrix} 1 & 0 \\ 0 & \alpha^{-1} \end{pmatrix} z)^{2k} f\left(\gamma_i \begin{pmatrix} 1 & 0 \\ 0 & \alpha^{-1} \end{pmatrix} z\right).$$

Note that unless $\deg \alpha$ is even, $\begin{pmatrix} 1 & 0 \\ 0 & \alpha^{-1} \end{pmatrix} z \notin \mathfrak{H}$. One easily verifies that the action of these Hecke operators on Foureir coefficients is analogous to the action of classical Hecke operators.

§3. **Main results.** Let $\alpha \in \mathbb{A}$ and $m \in \mathbb{Z}$. We let $r_k(\alpha, m)$ denote the restricted representation number

$$r_k(\alpha, m) = \#\left\{(\beta_1, \ldots, \beta_k) \in \mathbb{A}^k : \deg \beta_j < m, \sum_j \beta_j^2 = \alpha\right\}.$$

Clearly $r_k(\alpha, m) = 0$ if $\deg \alpha \geq 2m - 1$. Note that

$$(\theta(z))^k = \sum_{\alpha \in \mathbb{A}} r_k(\alpha, m) e\left\{T^2 \alpha x\right\}$$

where $z = \begin{pmatrix} T^{-2m} & x \\ 0 & 1 \end{pmatrix}$ (that is, $r_k(\alpha, m) = c_{T^2\alpha}(T^{-2m})$). Thus to study the representation numbers we will study the automorphic form $(\theta(z))^k$. However, since $\theta(z)^k$ has weight $\frac{k}{2}$, we will ease our computations here by only considering even powers of $\theta(z)$. Thus we fix $k \in \mathbb{Z}_+$, and we consider the weight k automorphic form $\theta(z)^{2k}$. Also, to ease the notation, let $r(\alpha, m) = r_{2k}(\alpha, m)$.

As we mentioned earlier, we will evenually analyze the Fourier coefficents of $\theta(z)^{2k}$ by anlayzing its behavior on a fundamental domain. To prepare for this we have

Proposition 3.1.
(1) The set $\left\{ \begin{pmatrix} T^{2m} & x \\ 0 & 1 \end{pmatrix} : \ m \geq 0, \ x \in T^{2m+1}\mathbb{A}/\mathbb{A} \right\}$ *is a complete set of coset representatives for* $\Gamma_\infty \backslash \mathfrak{H}$.
(2) The set $\left\{ \begin{pmatrix} T^{2m} & 0 \\ 0 & 1 \end{pmatrix} : \ m \geq 0 \right\}$ *is a complete set of coset representatives for* $\Gamma \backslash \mathfrak{H}$.

Proof. The proof of (1) is trivial and hence is omitted. The proof of (2) follows easily from the observations that $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : \ a \in \mathbb{A} \right\}$ generate Γ, and for $z = \begin{pmatrix} T^{-2m} & x \\ 0 & 1 \end{pmatrix} \in \mathfrak{H}$, $x \in T^{1-2m}\mathbb{A}$,

$$-\frac{1}{z} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} z = \begin{cases} \begin{pmatrix} T^{2m} & 0 \\ 0 & 1 \end{pmatrix} & \text{if } x = 0, \\ \begin{pmatrix} T^{2n-2m} & -\frac{1}{x} \\ 0 & 1 \end{pmatrix} & \text{if } x \neq 0, \deg x = -n. \end{cases}$$

□

Next we need the following technical Lemma.

Lemma 3.2. *Let* $d \in \mathbb{Z}_+$, $u = \sum_{-d<j\leq 0} u_j T^j$ *with* $u_0 \neq 0$. *Write*

$$\frac{1}{u} = \sum_{-d<j\leq 0} v_j T^j + \tilde{v}$$

where $\tilde{v} \in \mathfrak{P}_\infty^d$. *Then for* $0 \leq n < d$, v_{-n} *depends only on* $u_0, u_{-1}, \ldots, u_{-n}$, *and it is linear in* u_{-n}.

Proof. We argue by induction on n. One easily sees that $v_0 = \frac{1}{u_0}$, so assume $0 < n < d$, and suppose that v_j depends only on $u_0, \ldots, u_j$ for $-n < j \leq 0$. Set

$$u' = \sum_{-n<j\leq 0} u_j T^j, \quad \textit{and} \quad v' = \sum_{-n<j\leq 0} v_j T^j.$$

Then $u = u' + u_{-n}T^{-n} + u''$, and $\frac{1}{u} = v' + v_{-n}T^{-n} + v''$ where $u'', v'' \in \mathfrak{P}_\infty^{n+1}$. Thus

$$1 = u'v' + (u_0 v_{-D} + v_0 u_{-n})T^{-n} + w$$

where $w \in \mathfrak{P}_\infty^{n+1}$. So $v_{-n} = -\frac{1}{u_0}(v_0 u_{-n} + [u'v']_{-n})$ where $[u'v']_{-n}$ denotes the coefficient of T^{-n} in the expansion of $u'v'$. By hypothesis, v' depends only on $u_0, \dots, u_{1-n}$; thus v_{-n} depends only on $u_0, \dots, u_{-n}$ and is linear in u_{-n}. □

With this we get our preliminary description of the 0th Fourier coefficients, or equivalently, the number of ways to represent 0 as a sum of $2k$ squares.

Proposition 3.3. *For $m \in \mathbb{Z}_+$,*

$$r(0,m) = p^{2m(k-1)+1} + (p-1) \sum_{1 \le n < 2m} p^{n(k-1)} (-1|p)^{nk} r(0, m-n).$$

Proof. We show that

$$c_0(T^{-2m}) = p^{2m(k-1)+1} + (p-1) \sum_{1 \le n < 2m} p^{n(k-1)} (-1|p)^{nk} c_0(T^{2n-2m}).$$

We know from Theorem 1.3 that

$$\begin{aligned} &p^{2m-1} c_0(T^{-2m}) \\ &\quad = \sum_{x \in T^{1-2m}\mathbb{A}/\mathbb{A}} f\left(\begin{pmatrix} T^{-2m} & x \\ 0 & 1 \end{pmatrix}\right) \\ &\quad = f\left(\begin{pmatrix} T^{-2m} & 0 \\ 0 & 1 \end{pmatrix}\right) \\ &\qquad + \sum_{1 \le n < 2m} \sum_{u \in \left(\mathcal{O}_\infty/\mathfrak{P}_\infty^{2m-n}\right)^\times} f\left(\begin{pmatrix} T^{-2m} & T^{-n}u \\ 0 & 1 \end{pmatrix}\right) \end{aligned}$$

and applying the inversion formula (Theorem 2.3),

$$\begin{aligned} &= p^{2mk} f\left(\begin{pmatrix} T^{2m} & 0 \\ 0 & 1 \end{pmatrix}\right) \\ &\quad + \sum_{1 \le n < 2m} p^{nk}(-1|p)^{nk} \sum_{u \in \left(\mathcal{O}_\infty/\mathfrak{P}_\infty^{2m-n}\right)^\times} f\left(\begin{pmatrix} T^{2n-2m} & T^n \frac{1}{u} \\ 0 & 1 \end{pmatrix}\right). \end{aligned}$$

We know that $f\left(\begin{pmatrix} T^{2m} & 0 \\ 0 & 1 \end{pmatrix}\right) = c_0(T^{2m}) = 1$; similarly, for $m \le n < 2m$,

$$f\left(\begin{pmatrix} T^{2n-2m} & * \\ 0 & 1 \end{pmatrix}\right) = c_0(T^{2n-2m}) = 1.$$

Note that $\left(\mathcal{O}_\infty/\mathfrak{P}_\infty^{2m-n}\right)^\times$ has $(p-1)p^{2m-n-1}$ elements. Thus

$$\begin{aligned} p^{2m-1} c_0(T^{-2m}) = p^{2mk} c_0(T^{2m}) \\ &+ p^{2m-1}(p-1) \sum_{m \le n < 2m} p^{n(k-1)} (-1|p)^{nk} c_0(T^{2n-2m}) \\ &+ \sum_{1 \le n < m} \sum_{u} f\left(\begin{pmatrix} T^{2n-2m} & T^n \frac{1}{u} \\ 0 & 1 \end{pmatrix}\right). \end{aligned}$$

If $m = 1$ then we are done. So suppose $m > 1$, and fix n, $1 \le n < m$, and choose $u \in \left(\mathcal{O}_\infty/\mathfrak{P}_\infty^{2m-n}\right)^\times$. From the previous Lemma we know that $\frac{1}{u}\sum_j \tilde{u}_j T^j$ where, for $n - 2m < j \le -1$, $\tilde{u}_{-j}$ is dependent only on $u_0, \ldots, u_j$ and it is linear in u_j. Recalling that f is invariant under the left action of Γ_∞ and the right action of $PSL_2(\mathcal{O}_\infty)$, we see that

$$f\left(\begin{pmatrix} T^{2n-2m} & T^n \frac{1}{u} \\ 0 & 1 \end{pmatrix}\right) = f\left(\begin{pmatrix} T^{2n-2m} & v \\ 0 & 1 \end{pmatrix}\right)$$

with $v = \sum_{2n-2m<j<0} \tilde{u}_{j+n} T^j$. As u varies over $\left(\mathcal{O}_\infty/\mathfrak{P}_\infty^{2m-n}\right)^\times$, the preceding Lemma implies that the corresponding v are evenly distributed over $\mathfrak{P}_\infty/\mathfrak{P}_\infty^{2m-2n}$. Thus

$$\begin{aligned} &p^{2m-1} c_0(T^{-2m}) \\ &\quad = p^{2mk} + p^{2m-1}(p-1) \sum_{m \le n < 2m} p^{n(k-1)} (-1|p)^{nk} c_0(T^{2n-2m}) \\ &\qquad + \sum_{1 \le n < m} p^{nk} (-1|P)^{nk} (p-1) p^n \sum_{v \in \mathfrak{P}_\infty/\mathfrak{P}_\infty^{2n-2m}} f\left(\begin{pmatrix} T^{2n-2m} & v \\ 0 & 1 \end{pmatrix}\right). \end{aligned}$$

Now, Theorem 1.3 shows that

$$\begin{aligned} \sum_{v \in \mathfrak{P}_\infty/\mathfrak{P}_\infty^{2n-2m}} f\left(\begin{pmatrix} T^{2n-2m} & v \\ 0 & 1 \end{pmatrix}\right) &= \sum_{v \in T\mathbb{A}/T^{1+2n-2m}\mathbb{A}} f\left(\begin{pmatrix} T^{2n-2m} & v \\ 0 & 1 \end{pmatrix}\right) \\ &= p^{2m-2n-1} c_0(T^{2n-2m}), \end{aligned}$$

and the Theorem now follows. □

Our next goal is to describe the representation numbers in closed form. Once again, we first need a Lemma.

Lemma 3.4. *For $m \ge 1$,*

$$r(0, m) = (-1|p)^k p^k r(0, m-1) + p^{2m(k-1)+1} - (-1|p)^k p^{(2m-1)(k-1)}.$$

Proof. We use induction on m to show that

$$c_0(T^{-2m}) = (-1|p)^k p^k c_0(T^{-2(m-1)}) + p^{2m(k-1)+1} - (-1|p)^k p^{(2m-1)(k-1)}.$$

Recall that by Proposition 3.3,

$$c_0(T^{-2m}) = p^{2m(k-1)+1} + (p-1) \sum_{1 \le n < 2m} p^{n(k-1)} (-1|p)^{nk} c_0(T^{2n-2m}).$$

Case $m = 1$ of the Lemma follows by noting that $p^{2(k-1)+1} + (p-1)p^{(k-1)}(-1|p)^k = (-1|p)^k p^k + p^{2(k-1)+1} - (-1|p)^k p^{(k-1)}$. Now assume the statement of the Lemma is true for $m < M$, and write

$$\begin{aligned} c_0(T^{-2M}) =& p^{2M(k-1)+1} + (p-1) \sum_{1\le n<M} p^{n(k-1)}(-1|p)^{nk} c_0(T^{2n-2M}) \\ &+ (p-1) \sum_{M\le n<2M} p^{n(k-1)}(-1|p)^{nk} \\ =& p^{2M(k-1)+1} \\ &+ (p-1)\Big(\sum_{1\le n<M} p^{n(k-1)+k}(-1|p)^{(n+1)k} c_0(T^{-2(M-n-1)}) \\ &+ \sum_{M\le n<2M-1} \big(p^{(n+1)(k-1)+1}(-1|p)^{(n+1)k} - (-1|p)^{nk}p^{n(k-1)}\big) \\ &+ \sum_{M\le n<2M} p^{n(k-1)}(-1|p)^{nk}\Big) \end{aligned}$$

$$\begin{aligned} =& p^{2M(k-1)+1} + (-1|p)^k p^k c_0(T^{-2(M-1)}) - (-1|p)^k p^{2(M-1)(k-1)+1+k} \\ &+ (p-1)\big(p^{(2M-1)(k-1)+1}(-1|p)^k + p^{(2M-1)(k-1)}(-1|p)^k\big) \\ =& p^{2M(k-1)+1} + (-1|p)^k p^k c_0(T^{-2(M-1)}) - (-1|p)^k p^{(2M-1)(k-1)}. \end{aligned}$$

Now we have

Theorem 3.5. *For $k = 2$, $m > 0$,*

$$r(0, m) = mp^{2m+1} + p^{2m} - mp^{2m-1};$$

for $k \neq 2$, $m > 0$,

$$r(0, m) = p^{2m(k-1)+1} + (p-1)(p^k y_m + (-1|p)^k y_{m+1})$$

where $y_m = \begin{cases} (p+1)p^{mk-3}\left(\frac{1-p^{(m-1)(k-2)}}{1-p^{2(k-2)}}\right) & \text{if } m \text{ is odd} \\ (p+1)p^{(m+1)k-5}\left(\frac{1-p^{(m-2)(k-2)}}{1-p^{2(k-2)}}\right) + p^{(m-1)k-1} & \text{if } m \text{ is even.} \end{cases}$

Proof. We again use induction on m; again recall that $r(0, m) = c_0(T^{-2m})$. Applying Proposition 3.3, we see that the case $m = 1$, $k \neq 2$ is simply the statement that $p^{k-1}(-1|p)^k = p^k y_1 + (-1|p)^k y_2$. For $m = 1, k = 2$, again use Proposition 3.3 to see that $c_0(T^{-2m}) = p^3 + (p-1)p$ as required.

Now suppose the statement of the Theorem is established for $m < M$. For $k = 2$, applying the induction hypothesis and Lemma 3.4, we see that $c_0(T^{-2M}) = p^2 c_0(T^{-2(M-1)}) + p^{2M-1} - p^{2M+1} = Mp^{2M+1} + p^{2M} - Mp^{2M-1}$.

For $k \neq 2$, we use Lemma 3.4 to write

$$\begin{aligned} c_0(T^{-2M}) =& p^{2M(k-1)+1} - (-1|p)^k p^{(2M-1)(k-1)} + (-1|p)^k p^k c_0(T^{-2(M-1)}) \\ =& p^{2M(k-1)+1} + (-1|p)^k\big(- p^{(2M-1)(k-1)} + p^{(2M-1)(k-1)+2} \\ &+ p^{2k}(p-1)y_{M-1}\big) + p^k(p-1)y_M \end{aligned}$$

The Theorem follows by checking that our definitions for y_M satisfy $y_{M+1} = p^{2k}y_{M-1} + (p+1)p^{(2M-1)(k-1)}$. □

Next, we want to describe the restricted representation number $r(\alpha, m)$ where $\alpha \in \mathbb{A}$ is nonzero. As when $\alpha = 0$, we first find a recursive formula. Unfortunately, when $\deg \alpha > 3$, this formula involves Kloosterman sums (which we define shortly). Still, this formula leads to a simple description of the average number of ways to represent a polynomial of fixed degree, and we are able to describe the behavior of $r(\alpha, m)$ as $k \to \infty$, as $m \to \infty$, or as $\deg \alpha \to \infty$ (see Theorems 3.11, 3.14, 3.15).

We now need two simple Lemmas.

Lemma 3.6. *Let $\alpha \in \mathbb{K}_\infty$, $d \in \mathbb{Z}_+$ such that $D = \deg \alpha \leq d$. Then*

$$\sum_{u \in (\mathcal{O}_\infty/\mathfrak{P}_\infty^d)^\times} e\{\alpha u\} = \begin{cases} (p-1)p^{d-1} & \text{if } \alpha \in \mathcal{O}_\infty, \\ -p^{d-1} & \text{if } D = 1, \\ 0 & \text{if } D > 1. \end{cases}$$

Proof. First suppose $\alpha \in \mathcal{O}_\infty$. Then $e\{\alpha u\} = 1$ for all $u \in \mathcal{O}_\infty$, so $\sum_u e\{\alpha u\}$ is equal to the cardinality of $\left(\mathcal{O}_\infty/\mathfrak{P}_\infty^d\right)^\times$.

Next, suppose $D = 1$. So $\alpha = \alpha_1 T + \alpha'$, $\alpha' \in O$, and

$$\begin{aligned} \sum_u e\{\alpha u\} &= \sum_u e\{\alpha_1 T^u\} \\ &= \sum_u e\{\alpha_1 u_0 T\} \\ &= p^{d-1} \sum_{u_0 \in \mathbb{F}^\times} e\{\alpha_1 u_0 T\} \\ &= -p^{d-1}. \end{aligned}$$

Finally, suppose $D > 1$. So $\alpha = \alpha_D T^D + \cdots + \alpha_1 T + \alpha'$, $\alpha' \in \mathcal{O}_\infty$. Thus

$$\begin{aligned} \sum_u e\{\alpha u\} &= \sum_{\substack{u' \in \left(\mathcal{O}_\infty/\mathfrak{P}_\infty^{D-1}\right)^\times \\ u'' \in \mathfrak{P}_\infty^{D-1}/P^d}} e\{\alpha(u' + u'')\} \\ &= \sum_{u'} e\{\alpha u'\} \, p^{d-D} \sum_{u_{1-D} \in \mathbb{F}} e\{\alpha_D u_{1-D} T\} \\ &= 0 \end{aligned}$$

since the sum on u_{1-D} is a nontrivial character sum. □

Let $d \in \mathbb{Z}_+$, $\alpha, \beta \in \mathbb{K}_\infty$ such that $\deg \alpha, \deg \beta \leq d$. Then we define the Kloosterman sum $K(\alpha, \beta; \mathfrak{P}_\infty^d)$ by

$$K(\alpha, \beta; \mathfrak{P}_\infty^d) = \sum_{u \in (\mathcal{O}_\infty/\mathfrak{P}_\infty^d)^\times} e\left\{\alpha u + \frac{\beta}{u}\right\}.$$

Since $\deg \alpha, \deg \beta \leq d$, the sum on u is well-defined. Notice that $u \mapsto \frac{1}{u}$ is an automorphism of $\left(\mathcal{O}_\infty/\mathfrak{P}_\infty^d\right)^\times$, so $K(\alpha, \beta; \mathfrak{P}_\infty^d) = K(\beta, \alpha; \mathfrak{P}_\infty^d)$. Also note that if $\alpha - \alpha' \in \mathcal{O}_\infty$ then $K(\alpha', \beta; \mathfrak{P}_\infty^d) = K(\alpha, \beta; \mathfrak{P}_\infty^d)$, and if $d \geq D = \max\{\deg \alpha, \deg \beta\}$ then

$$K(\alpha, \beta; \mathfrak{P}_\infty^d) = p^{d-D} K(\alpha, \beta; \mathfrak{P}_\infty^D)$$

(since $e\left\{\alpha w + \frac{\beta}{w}\right\} = 1$ for $w \in \mathfrak{P}_\infty^D$).

Lemma 3.7. *Let* $\alpha, \beta \in \mathbb{K}_\infty$ *and* $d \in \mathbb{Z}_+$ *such that* $\deg \beta \le \deg \alpha \le d$. *Then*

$$K(\alpha, \beta; \mathfrak{P}_\infty^d) = \begin{cases} (p-1)p^{d-1} & \text{if } \alpha, \beta \in \mathcal{O}_\infty, \\ -p^{d-1} & \text{if } \beta \in \mathcal{O}_\infty,\ \deg \alpha = 1, \\ 0 & \text{if } \deg \alpha > 1,\ \deg \alpha \ne \deg \beta. \end{cases}$$

Proof. If $\beta \in \mathcal{O}_\infty$ then $K(\alpha, \beta; \mathfrak{P}_\infty^d) = \sum_{u \in (\mathcal{O}_\infty/\mathfrak{P}_\infty^d)^\times} e\{\alpha u\}$, so the Lemma follows from the preceding Lemma. So suppose $\beta \notin \mathcal{O}_\infty$ and $D = \deg \alpha > \deg \beta$. Then $\alpha = \alpha_D T^D + \cdots + \alpha_1 T + v$ and $\beta = \beta_{D-1} T^{D-1} + \cdots + \beta_1 T + w$ where $v, w \in \mathcal{O}_\infty$ and $\alpha_D \ne 0$. So

$$K(\alpha, \beta; \mathfrak{P}_\infty^d) = \sum_{u_0, \dots, u_{1-d}} e\{T(\alpha_D u_{1-D} + \cdots + \alpha_1 u_0 + \beta_{D-1} \tilde{u}_{2-D} + \cdots + \beta_1 \tilde{u}_0)\};$$

here $u_0, \dots, u_{1-d}$ vary over $\mathbb{F}$, $u_0 \ne 0$, and $\frac{1}{u} = \sum_j \tilde{u}_j T^j$. We know that $\tilde{u}_0, \dots, \tilde{u}_{2-D}$ are independent of u_{1-D}. Hence we can isolate the sum on u_{1-D}; since $\alpha_D \ne 0$, we have a nontrivial character sum. So $K(\alpha, \beta; \mathfrak{P}_\infty^d) = 0$ in this case. □

Now we obtain our preliminary description of $r(\alpha, m)$ for $\alpha \ne 0$.

Proposition 3.8. *Take* $\alpha \in \mathbb{A}$, $\alpha \ne 0$; *let* $D = \deg \alpha$ *and fix* $m \in \mathbb{Z}_+$ *such that* $D < 2m - 1$. *Then*

$$\begin{aligned} r(\alpha, m) &\\ = p \cdot &\sum_{D+1 < n \le 2m} p^{n(k-1)} (-1|p)^{nk} r(0, m-n) \\ - &\sum_{D+1 \le n < 2m} p^{n(k-1)} (-1|p)^{nk} r(0, m-n) \\ + &\sum_{1 \le n \le \frac{D}{2}} p^{nk} (-1|p)^{nk} \sum_{\substack{\beta \in \mathbb{A} \\ \deg \beta = D - 2n}} r(\beta, m-n) p^{-D-2} K(\alpha T^{2-n}, \beta T^{2+n}; \mathfrak{P}_\infty^{D+2-n}). \end{aligned}$$

Proof. We prove the analogous formula for $c_{T^2\alpha}(T^{-2m})$. Take $\alpha \in T^2\mathbb{A}$, $\alpha \ne 0$; let $D = \deg \alpha$. Fix $m \in \mathbb{Z}_+$ such that $D \le 2m$. Recall that with $z = \begin{pmatrix} T^{-2m} & x \\ 0 & 1 \end{pmatrix}$, we have

$$-\frac{1}{z} = \begin{cases} \begin{pmatrix} T^{2m} & 0 \\ 0 & 1 \end{pmatrix} & \text{if } x = 0, \\ \begin{pmatrix} T^{2n-2m} & -\frac{1}{x} \\ 0 & 1 \end{pmatrix} & \text{if } x \ne 0,\ -n = \deg x > -2m. \end{cases}$$

Also note that $T^{-n}u + \mathbb{A}$ runs over the nonzero elements of $T^{2m+1}\mathbb{A}/\mathbb{A}$ as n and u vary, $1 \le n < 2m$, $u \in \left(\mathcal{O}_\infty/\mathfrak{P}_\infty^{2m-n}\right)^\times$. Thus the Fourier transform and the inversion formula give us

$$\begin{aligned} p^{2m-1} c_\alpha(T^{-2m}) &\\ = p^{2mk} f &\left(\begin{pmatrix} T^{2m} & 0 \\ 0 & 1 \end{pmatrix} \right) \\ &+ \sum_{1 \le n < 2m} p^{nk} (-1|p)^{nk} \sum_{u \in \left(\mathcal{O}_\infty/\mathfrak{P}_\infty^{2m-n}\right)^\times} f\left(\begin{pmatrix} T^{2n-2m} & T^n \frac{1}{u} \\ 0 & 1 \end{pmatrix} \right) e\{\alpha T^{-n} u\} \end{aligned}$$

and since $f\left(\begin{pmatrix} T^{2n-2m} & * \\ 0 & 1 \end{pmatrix}\right) = c_0(T^{2n-2m}) = 1$ when $n \geq m$,

$$= p^{2mk} + \sum_{m \leq n < 2m} p^{nk}(-1|p)^{nk} c_0(T^{2n-2m}) \cdot \sum_{u \in \left(\mathcal{O}_\infty/\mathfrak{P}_\infty^{2m-n}\right)^\times} e\left\{\alpha T^{-n} u\right\}$$

$$+ \sum_{1 \leq n < m} p^{nk}(-1|p)^{nk} \sum_{\substack{u \in \left(\mathcal{O}_\infty/\mathfrak{P}_\infty^{2m-n}\right)^\times \\ \beta \in T^2\mathbb{A}}} c_\beta(T^{2n-2m}) e\left\{\alpha T^{-n} u + \beta T^n/u\right\}.$$

Now by Lemma 3.7 and Proposition 3.8, this is equal to

$$p^{2mk} + \sum_{D \leq n < 2m} p^{nk}(-1|p)^{nk}(p-1)p^{2m-n-1} c_0(T^{2n-2m})$$

$$- p^{(D-1)k}(-1|p)^{(D-1)k} p^{2m-D} c_0(T^{2n-2m})$$

$$+ \sum_{\substack{1 \leq n < m \\ \deg \beta = D-2n}} p^{nk}(-1|p)^{nk} c_\beta(T^{2n-2m}) K(\alpha T^{-n}, \beta T^n; \mathfrak{P}_\infty^{2n}$$

Finally, we recall that since $\deg \alpha T^{-n} = \deg \beta T^n = D-n$, $K(\alpha T^{-n}, \beta T^n; \mathfrak{P}_\infty^{2m-n}) = p^{2m-D} K(\alpha T^{-n}, \beta T^n; \mathfrak{P}_\infty^{D-n})$.□

Next we remove some of the recursion for our formula for $r(\alpha, m)$ by analyzing what we consider the main term.

Theorem 3.9. *Suppose $\alpha \in \mathbb{A}$ is nonzero with $\deg \alpha = D < 2m-1$. Write*

$$r(\alpha, m) = g_{D,m} + \sum_{\substack{1 \leq n \leq \frac{D}{2} \\ \deg \beta = D-2n}} p^{nk}(-1|p)^{nk} r(\beta, m-n) p^{-D-2} K(\alpha T^{2-n}, \beta T^{2+n}; \mathfrak{P}_\infty^{D+2-n}).$$

If $k = 1$, we have

$$g_{D,m} = \left(\left[\frac{D+2}{2}\right] p - \left[\frac{D+1}{2}\right] \right) + (-1|p)\left(\left[\frac{D+1}{2}\right] p - \left[\frac{D+2}{2}\right] \right).$$

If $k \neq 1$, then

$$g_{D,m} = p^{2m(k-1)+1} + s_{D,m} + (-1|p)^k p^{k-1} s_{D+1,m}$$

where $s_{D,m} = \begin{cases} (p-1)p^{(2m-D)(k-1)} \frac{(1-p^{(k-1)D})}{1-p^{2(k-1)}} & \textit{if } D \textit{ is even} \\ (p-1)p^{(2m-D+1)(k-1)} \frac{(1-p^{(k-1)(D-1)})}{(1-p^{2(k-1)})} - p^{(2m-D-1)(k-1)} & \textit{if } D \textit{ is odd.} \end{cases}$

Proof. Again, we prove the analogous formula for $c_\alpha(T^{-2m})$ where $\alpha \in T^2\mathbb{A}, \alpha \neq 0$, and $D = \deg \alpha$. From Proposition 3.8, we know that

$$\begin{aligned} g_{D,m} =& p^{2m(k-1)+1} + \sum_{D \leq n < 2m} p^{n(k-1)}(-1|p)^{nk}(p-1)c_0(T^{2n-2m}) \\ &- p^{(D-1)(k-1)}(-1|p)^{(D-1)k} c_0(T^{2(D-1)-2m}) \\ =& c_0(T^{-2m}) - \sum_{1 \leq n < D} p^{n(k-1)}(-1|p)^{nk}(p-1)c_0(T^{2n-2m}) \\ &- p^{(D-1)(k-1)}(-1|p)^{(D-1)k} c_0(T^{2(D-1)-2m}) \end{aligned}$$

We will use induction on D and Lemma 3.4 to show that $g(2m)$ has the required form. If $D = 2$, we see that

$$g(2m) = c_0(T^{-2m}) - p^k(-1|p)^k c_0(T^{-2m+2}) = p^{2m(k-1)+1} - (-1|p)^k p^{(2m-1)(k-1)}.$$

If $k = 1$, this reduces to $p - (-1|p)$.

Now we assume that the statement of the Theorem is true for $D < d$, and note that if $g_{D,m}$ is associated with an α of degree d-1 while $g_{d,m}$ is associated with an α' of degree d, then

$$\begin{aligned} g_{d,m} =& g_{D,m} + p^{(d-2)(k-1)}(-1|p)^{(d-2)k} c_0(T^{-2(m-d+2)}) \\ & - p^{(d-1)(k-1)+1}(-1|p)^{(d-1)k} c_0(T^{-2(m-d+1)}) \\ =& g_{D,m} + p^{(d-2)(k-1)}(-1|p)^{(d-2)k}\big(p^{(2m-2d+4)(k-1)+1} \\ & - (-1|p)^k p^{(2m-2d+3)(k-1)}\big) \end{aligned}$$

In the case $k = 1$, the induction hypothesis then shows that

$$\begin{aligned} g_{d,m} =& \Big(\left[\frac{d-1}{2}\right] p - \left[\frac{d-2}{2}\right] \Big) + (-1|p)\Big(\left[\frac{d-2}{2}\right] p - \left[\frac{d-1}{2}\right] \Big) + (-1|p)^d (p - (-1|p)) \\ =& \Big(\left[\frac{d}{2}\right] p - \left[\frac{d-1}{2}\right] \Big) + (-1|p)\Big(\left[\frac{d-1}{2}\right] p - \left[\frac{d}{2}\right] \Big). \end{aligned}$$

If $k \neq 1$,

$$g'(2m) = \begin{cases} s_{d-1,m} + p^{(2m-d+2)(k-1)+1} + (-1|p)^k (p^{k-1} s_{d,m} - p^{(2m-d+1)(k-1)}) & \text{if } d \text{ is even,} \\ s_{d-1,m} + p^{(2m-d+1)(k-1)} + (-1|p)^k (p^{k-1} s_{d,m} + p^{(2m-d+2)(k-1)+1}) & \text{if } d \text{ is odd.} \end{cases}$$

It will suffice to show that our formulas for $s_{d,m}$ satisfy

$$s_{d,m} = \begin{cases} s_{d-1,m} + p^{(2m-d+2)(k-1)} & \text{if } d \text{ is even,} \\ s_{d-1,m} - p^{(2m-d+1)(k-1)} & \text{if } d \text{ is odd.} \end{cases}$$

This is easily verified. □

We consider here a few special cases.

Corollary 3.10. *For $\alpha \in \mathbb{A}$, let R_α denote the number of distinct nonzero roots of α which lie in $\mathbb{F}$; let $g(2m)$ be as in the preceding Theorem.*

(1) Suppose $\alpha \in \mathbb{A}$ with $\deg \alpha = 0$ or 1. Then for $m \in \mathbb{Z}_+$ with $D = \deg \alpha < 2m-1$, $r(\alpha, m) = g(2m)$.

(2) Suppose $\alpha \in \mathbb{A}$ has degree 2. Then for $m \in \mathbb{Z}_+$ with $m \geq 2$,

$$r(\alpha, m) = g_{2,m} + (-1|p)^k g_{0,m} p^{k-3}(pR_\alpha - p + 1).$$

(3) Suppose $\alpha \in \mathbb{A}$ has degree 3. Then for $m \in \mathbb{Z}_+$ with $m \geq 3$,

$$r(\alpha, m) = g_{3,m} + (-1|p)^k g_{1,m} p^{k-4}(pR_\alpha - p + 1).$$

CHECK THESE FOR SMALL M?

Proof. Recall that $r(\alpha, m) = c_{T^2\alpha}(T^{-2m})$; thus we analyze $c_\alpha(T^{-2m})$ where $\alpha \in T^2\mathbb{A}$ has degree 4 or 5.

(1) This follows immediately from the preceding Theorem.

(2) First suppose $\alpha, \beta \in T^2\mathbb{A}$ with $\deg \alpha = 4$ and $\deg \beta = 2$. Then for $u = \sum_j u_j T^j \in \mathcal{O}_\infty^\times$, we have

$$\frac{1}{u} = \frac{1}{u_0} - \frac{u_{-1}}{u_0^2} T^{-1} + \frac{u_{-1}^2 - u_0 u_{-2}}{u_0^3} T^{-2} + w$$

where $w \in \mathfrak{P}_\infty^3$. Thus

$$K(\alpha T^{-1}, \beta T; \mathfrak{P}_\infty^3) \\ = \sum_{u_0, u_{-1}, u_{-2}} e\left\{ T(\alpha_4 u_{-2} + \alpha_3 u_{-1} + \alpha_2 u_0 + \beta_2 \frac{u_{-1}^2 - u_0 u_{-2}}{u_0^3}) \right\}.$$

Isolating the sum on u_{-2}, we have a character sum which yields 0 except when $\beta_2 = u_0^2 \alpha_4$. So $K(\alpha T^{-1}, \beta T; \mathfrak{P}_\infty^3) = 0$ unless $(\beta_2|p) = (\alpha_4|p)$. Suppose we have $(\beta_2|p) = (\alpha_4|p)$; then $\beta_2 = v_0^2 \alpha_4 = (-v_0)^2 \alpha_4$ for some $v_0 \in \mathbb{F}^\times$, and

$$K(\alpha T^{-1}, \beta T; \mathfrak{P}_\infty^3) = p \sum_{u_{-1} \in \mathbb{F}} e\left\{ T(\alpha_4 \frac{u_{-1}^2}{v_o} + \alpha_3 u_{-1} + \alpha_2 v_0) \right\} \\ + p \sum_{u_{-1} \in \mathbb{F}} e\left\{ T(-\alpha_4 \frac{u_{-1}^2}{v_0} + \alpha_3 u_{-1} - \alpha_2 v_0) \right\}$$

and replacing u_{-1} by $v_0 u_{-1}$ in the first sum and by $-v_0 u_{-1}$ in the second sum,

$$= p \sum_{u_{-1}} e\left\{ T v_0 (\alpha_4 u_{-1}^2 + \alpha_3 u_{-1} + \alpha_2) \right\} \\ + p \sum_{u_{-1}} e\left\{ -T v_0 (\alpha_4 u_{-1}^2 + \alpha_3 u_{-1} + \alpha_2) \right\}.$$

Now we sum over $\beta \in T^2\mathbb{A}$, $\deg \beta = 2$. (Note that $r(\beta, m-1) = g_{0,m-1}$ since $\deg \beta = 1$.) So this means we vary v_0^2 over $\mathbb{F}^\times$. Thus we have

$$\sum_{\substack{\beta \in T^2\mathbb{A} \\ \deg \beta = 2}} K(\alpha T^{-1}, \beta T; \mathfrak{P}_\infty^3) \\ = p \sum_{\substack{u_{-1} \in \mathbb{F}^\times \\ v_0 \in \mathbb{F}^\times}} e\left\{ T v_0 (\alpha_4 u_{-1}^2 + \alpha_3 u_{-1} + \alpha_2) \right\}.$$

The sum on v_0 is equal to $p - 1$ if u_{-1} is a root of $\alpha_2 + \alpha_3 T + \alpha_4 T^2$, and -1 otherwise. Thus

$$\sum_\beta K(\alpha T^{-1}, \beta T; \mathfrak{P}_\infty^3) = p(pR_\alpha - p + 1).$$

The result now follows from the preceding Theorem.

(3) Now let $\alpha = \alpha_5 T^5 + \alpha_4 T^4 + \alpha_3 T^3 + \alpha_2 T^2$ and $\beta = \beta_3 T^3 + \beta_2 T^2$. For $u = \sum_j u_j T^j \in \mathcal{O}_\infty^\times$, we have

$$\frac{1}{u} = \frac{1}{u_0} - \frac{u_{-1}}{u_0^2}T^{-1} + \frac{u_{-1}^2 - u_0 u_{-2}}{u_0^3}T^{-2} + \frac{2u_0u_{-1}u_{-2} - u_{-1}^3 - u_0^2 u_{-2}}{u_0^4}T^{-3} + w$$

where $w \in \mathfrak{P}_\infty^4$. So

$$\begin{aligned} &K(\alpha T^{-1}, \beta T; \mathfrak{P}_\infty^4) \\ &\quad = \sum_{u_0, \dots, u_{-3}} e\left\{T(\alpha_5 u_{-3} + \alpha_4 u_{-2} + \alpha_3 u_{-1} + \alpha_2 u_0\right\} \\ &\qquad \cdot e\left\{\beta_3 \frac{2u_0u_{-1}u_{-2} - u_{-1}^3 - u_0^2 u_{-2}}{u_0^4} + \beta_2 \frac{u_{-1}^2 - u_0 u_{-2}}{u_0^3})\right\}. \end{aligned}$$

The sum on u_{-3} is 0 if $\beta_3 \neq u_0^2 \alpha_5$, and p otherwise. So suppose $\beta_3 = v_0^2 \alpha_5 = (-v_0)^2 \alpha_5$ for some $v_0 \in \mathbb{F}^\times$. Then

$$\begin{aligned} &K(\alpha T^{-n}, \beta T^n; \mathfrak{P}_\infty^4) \\ &\quad = p \sum_{u_{-1}} e\left\{T(\alpha_3 u_{-1} + \alpha_2 v_0 - \frac{\alpha_5 u_{-1}^3}{v_0^2} + \frac{\beta_2 u_{-1}^2}{v_0^3})\right\} \\ &\qquad \cdot \sum_{u_{-2}} e\left\{T u_{-2}(\alpha_4 + \frac{2\alpha_5 u_{-1}}{v_0} - \frac{\beta_2}{v_0^2})\right\} \end{aligned}$$

and since the sum on u_{-2} is a character sum,

$$\begin{aligned} &= p^2 \sum_{u_{-1}} e\left\{T(\alpha_2 v_0 + \alpha_3 u_{-1} + \frac{\alpha_4 u_{-1}^2}{v_0} + \frac{\alpha_5 u_{-1}^3}{v_0^2})\right\} \\ &\quad + p^2 \sum_{u_{-1}} e\left\{T(-\alpha_2 v_0 + \alpha_3 u_{-1} - \frac{\alpha_4 u_{-1}^2}{v_0} + \frac{\alpha_5 u_{-1}^3}{v_0^2})\right\} \end{aligned}$$

and replacing u_{-1} by $v_0 u_{-1}$ in the first sum and by $-v_0 u_{-1}$ in the second sum,

$$\begin{aligned} &= p^2 \sum_{u_{-1}} e\left\{T v_0(\alpha_2 + \alpha_3 u_{-1} + \alpha_4 u_{-1}^2 + \alpha_5 u_{-1}^3)\right\} \\ &\quad + p^2 \sum_{u_{-1}} e\left\{-T v_0(\alpha_2 + \alpha_3 u_{-1} + \alpha_4 u_{-1}^2 + \alpha_5 u_{-1}^3)\right\}. \end{aligned}$$

Now we sum over β, so we need to vary v_0^2. (Note that $r(\beta, m-1) = g_{1,m-1}$ since $\deg \beta = 1$.) Thus we have

$$K(\alpha T^{-n}, \beta T^n; \mathfrak{P}_\infty^4) = \sum_{\substack{u_{-1} \in \mathbb{F} \\ v_0 \in \mathbb{F}^\times}} e\left\{T v_0(\alpha_2 + \alpha_3 u_{-1} + \alpha_4 u_{-1}^2 + \alpha_5 u_{-1}^3)\right\}.$$

The sum on v_0 yields $p-1$ if u_{-1} is a root of $T^{-2}\alpha$, and -1 otherwise. So we get

$$\sum_\beta K(\alpha T^{-1}, \beta T; \mathfrak{P}_\infty^4) = p^2(pR_\alpha + p - 1).$$

The result now follows from the preceding Theorem. □

This corollary suggests that the values of the sums of Kloosterman sums which appear in the formula in Theorem 2.9 vary wildly(?), depending on the factorization of α over $\mathbb{F}$. At present we are unable to analyze these sums of Kloosterman sums except in the cases treated above. However, we are able to determine the average value of these sums as we vary α as described in the next Theorem. We thank David Grant for suggesting this result.

Theorem 3.11. *Fix $D, m \in \mathbb{Z}_+$ such that $D < 2m-1$. For $\alpha \in \mathbb{A}$ with $\deg \alpha = D$, write $\alpha = \alpha_0 + \cdots + \alpha_D T^D$. Choose $t \in \mathbb{Z}$ such that $\left[\frac{D}{2}\right] \le t \le D$. Then, varying the coefficients $\alpha_0, \ldots, \alpha_t$ over $\mathbb{F}$, the average value of $r(\alpha, m)$ is*

$$\overline{r}(\alpha, m) = g_{D,m}.$$

Proof. We know $r(\alpha, m) = c_{T^2\alpha}(T^{-2m})$; thus by Theorem 3.9 it suffices to show that for any n, $1 \le n \le \frac{D}{2} + 1$ and $\beta \in T^2\mathbb{A}$ such that $\deg \beta = D + 2 - 2n$,

$$\sum_{\alpha_1, \ldots, \alpha_t} K(\alpha T^{2-n}, \beta T^n; \mathfrak{P}_\infty^{D+2-n}) = 0.$$

Note that the preceding corollary imples the Theorem holds for $D = 0$ or 1, so we may assume here that $D > 1$. Since $\left[\frac{D}{2}\right] \le t \le D$, α_{n-1} varies over $\mathbb{F}$; hence

$$\sum_{\alpha_1, \ldots, \alpha_t} K(\alpha T^{2-n}, \beta T^n; \mathfrak{P}_\infty^{D+2-n}) = \sum_{\substack{u,\ \alpha_j \\ j \ne n-1}} e\left\{\frac{\beta T^n}{u} + (\alpha - \alpha_{n-1}T^{n-1})T^{2-n}u\right\} \sum_{\alpha_{n-1}} e\left\{\alpha_{n-1}u_0 T\right\}$$

where $u = \sum_j u_j T^j \in \left(\mathcal{O}_\infty / \mathfrak{P}_\infty^{D-n}\right)^\times$, $\alpha_j \in \mathbb{F}$ (and $\alpha_D \ne 0$). Since $u_0 \ne 0$, the sum on α_{n-1} is a nontrivial character sum. □

Despite our poor understanding of sums of Kloosterman sums, Theorem 3.9 does allow us to determine the order of magnitude of $r(\alpha, m)$ as $m \to \infty$ or as $\deg \alpha \to \infty$, as well as an asymptotic formula for $r(\alpha, m)$ as $k \to \infty$. To obtain the first of these results, we need to find a bound for each sum of Kloosterman sums. Thus we have

Lemma 3.12. *Fix $\alpha \in T^2\mathbb{A}$ such that $\deg \alpha = D \ge 4$, and fix n, $1 \le n \le \frac{D}{2} - 1$. Then*

$$\left| \sum_{\substack{\beta \in T^2\mathbb{A} \\ \deg \beta = D - 2n}} K(\alpha T^{-n}, \beta T^n; \mathfrak{P}_\infty^{D-n}) \right| < p^{D-n}.$$

Proof. By definition,

$$\begin{aligned}
&\sum_\beta K(\alpha T^{-n},\beta T^n;\mathfrak{P}_\infty^{D-n})\\
&\quad=\sum_\beta \sum_{u\in(\mathcal{O}_\infty/\mathfrak{P}_\infty^{D-n})^\times} e\left\{\frac{\alpha T^{-n}}{u}+\beta T^n u\right\}\\
&\quad=\sum_{\substack{\beta_2,\dots,\beta_{D-2n}\\ u_0,\dots,u_{1+n-D}}} e\left\{T(\alpha_D\tilde u_{1+n-D}+\cdots+\alpha_{1+n}\tilde u_0+\beta_{D-2n}u_{1+n-D}+\cdots+\beta_2 u_{-n-1})\right\};
\end{aligned}$$

here $\beta_{D-2n}\neq 0$, $u_0\neq 0$, and $\frac{1}{u}=\sum_j \tilde u_j T^j$. As β_j varies over $\mathbb{F}$, $2\le j\le D-2n$, we get a character sum on β_j. So whenever $u_j\neq 0$, $2\le j\le D-2n$, the sum on the β_j becomes 0. Thus we are left with a sum on $u_0,\dots,u_{-n}$ and on u_{1+n-D}; this sum has $(p-1)p^{1+n}$ terms. Hence

$$\left|\sum_\beta K(\alpha T^{-n},\beta T^n;\mathfrak{P}_\infty^{D-n})\right| < p^{D-n}.$$

□

Lemma 3.13. *Let $g_{D,m}$ be as in Theorem 3.9. Then*

$$g_{D,m}=p^{2m(k-1)+1}+t_{D,m}$$

where $|t_{D,m}|<2p^{(2m02)(k-1)+1}(1+p^{k-1})$.

Proof. We know from Theorem 3.9 that

$$g_{D,m}=p^{2m(k-1)+1}+s_{D,m}+(-1|p)^k p^{k-1}s_{D+1,m}.$$

Now,

$$\begin{aligned}
&s_{D+1,m}\\
&=\begin{cases}(p-1)p^{(2m-D+1)(k-1)}(p^{(D-3)(k-1)}+p^{(D-5)(k-1)}+\cdots+1) & \text{if } D \text{ is odd,}\\ (p-1)p^{(2m-D+2)(k-1)}(p^{(D-4)(k-1)}+p^{(D-6)(k-1)}+\cdots+1)\\ \quad -p^{(2m-D)(k-1)} & \text{if } D \text{ is even,}\end{cases}\\
&=\begin{cases}(p-1)p^{(2m-2)(k-1)}(1+p^{-2(k-1)}+p^{-4(k-1)}+\cdots+p^{-(D-3)(k-1)}) & \text{if } D \text{ is odd,}\\ (p-1)p^{(2m-2)(k-1)}(1+p^{-2(k-1)}+p^{-4(k-1)}+\cdots+p^{-(D-4)(k-1)})\\ \quad -p^{(2m-D)(k-1)} & \text{if } D \text{ is even.}\end{cases}
\end{aligned}$$

So $|p^{k-1}s_{D+1,m}|<2p^{(2m-1)(k-1)+1}$, and $|s_{D,m}|<2p^{(2m-2)(k-1)+1}$. □

Theorem 3.14. *For $\alpha \in \mathbb{A}$, $k > 1$, $r(\alpha, m) = O(p^{2m(k-1)})$ as $m \to \infty$ or as $D = \deg \alpha \to \infty$ (with $2m = D + t$, t fixed).*

Proof. We show that for $\alpha \in T^2\mathbb{A}$, $c_\alpha(T^{-2m}) = O(p^{2m(k-1)})$ as $m \to \infty$ or as $D = \deg \alpha \to \infty$. So take $\alpha \in T^2\mathbb{A}$. To ease the notation, let $K_n(\alpha, \beta) = p^{n-D} K(\alpha T^{-n}, \beta T^n; \mathfrak{P}_\infty^{D-n})$ where $\deg \alpha = D$ and $\deg \beta = D - 2n$. Then using Theorem 3.9 repeatedly, we find that

$$
\begin{aligned}
&c_\alpha(T^{-2m}) \\
&\quad = g(2m) + \sum_{\substack{1 \le n_1 \le \frac{D}{2} - 1 \\ \deg \beta^{(1)} = D - 2n_1}} p^{n_1(k-1)} (-1|p)^{n_1 k} c_{\beta^{(1)}}(T^{2n_1 - 2m}) K_{n_1}(\alpha, \beta^{(1)}) \\
&\quad = g(2m) \\
&\qquad + \sum_{1 \le n_1 \le \frac{D}{2} - 1} p^{n_1(k-1)} (-1|p)^{n_1 k} g(2m - 2n_1) \sum_{\deg \beta^{(1)} = D - 2n_1} K_{n_1}(\alpha, \beta^{(1)}) \\
&\qquad + \sum_{\substack{1 \le n_1 \le \frac{D}{2} - 1 \\ 1 \le n_2 \le n_1}} p^{(n_1+n_2)(k-1)} (-1|p)^{(n_1+n_2)k} \sum_{\deg \beta^{(2)} = D - 2n_2} c_{\beta^{(2)}}(T^{2(n_1+n_2) - 2m}) \\
&\qquad\quad \cdot \sum_{\beta^{(1)}} K_{n_1}(\alpha, \beta^{(1)}) K_{n_2}(\alpha, \beta^{(2)}) \\
&\quad = g(2m) + \sum_{n_1} p^{n_1(k-1)} (-1|p)^{n_1 k} g(2m - 2n_1) \sum_{\beta^{(1)}} K_{n_1}(\alpha, \beta^{(1)}) \\
&\qquad + \sum_{n_1, n_2} p^{(n_1+n_2)(k-1)} (-1|p)^{(n_1+n_2)k} g(2m - 2(n_1 + n_2)) \\
&\qquad \cdot \sum_{\beta^{(1)}, \beta^{(2)}} K_{n_1}(\alpha, \beta^{(1)}) K_{n_2}(\beta^{(1)}, \beta^{(2)}) \\
&\qquad + \cdots + \\
&\qquad + \sum_{n_1, \dots, n_d} p^{(n_1 + \cdots + n_d)(k-1)} (-1|p)^{(n_1 + \cdots + n_d)k} g(2m - 2(n_1 + \cdots + n_d)) \\
&\qquad \cdot \sum_{\beta^{(1)}, \dots, \beta^{(d)}} K_{n_1}(\alpha, \beta^{(1)}) \cdots K_{n_d}(\beta^{(d-1)}, \beta^{(d)}) \qquad \blacksquare
\end{aligned}
$$

where $d = \left[\frac{D}{2} - 1\right]$. The previous Lemma shows that the sums on the $\beta^{(j)}$ are bounded by 1. Hence

$$
|c_\alpha(T^{-2m})| \le |g(2m)| + \sum_{1 \le r \le d} \sum_{n_1, \dots, n_r} p^{(n_1 + \cdots + n_r)(k-1)} \; |g(2m - 2(n_1 + \cdots + n_r))|
$$

where $1 \le n_1 \le \frac{D}{2} - 1$ and $1 \le n_j \le \frac{D}{2} - n_{j-1} - \cdots - n_1 - 1$ for $j > 1$. Now, fix n, $1 \le n \le \frac{D}{2} - 1$. Note that if $n_1 + \cdots + n_r = n$ for any positive integers n_j then we necessarily satisfy the above inequalities on the n_j. Thus, letting ρ be Ramanujan's partition function, we have

$$
|c_\alpha(T^{-2m})| \le |g(2m)| + \sum_{1 \le n \le \frac{D}{2} - 1} \rho(n) p^{n(k-1)} |g(2m - 2n)|
$$

and by the two preceding Lemmas,

$$\begin{aligned}
&\leq Bp^{2m(k-1)}\left(1+\sum_{1\leq n\leq \frac{D}{2}-1}\rho(n)p^{-n(k-1)}\right)\\
&\leq Bp^{2m(k-1)}\left(1+\sum_{1\leq n}\rho(n)p^{-n(k-1)}\right).
\end{aligned}$$

Now, Ramanujan showed (formula 8.3.3, p. 114, [Hardy]) that there is a nonzero constant B such that $\rho(n) < e^{B\sqrt{n}}$. Thus the series

$$\sum_{1\leq n}\rho(n)p^{-n(k-1)}$$

converges, and the Theorem is proved. □

Theorem 3.15. *Take* $\alpha\in\mathbb{A}$, $\alpha\neq 0$; *choose* $m\in\mathbb{Z}_+$ *such that* $\deg\alpha < 2m-1$. *Then as* $k\to\infty$,

$$r(\alpha,m) = p^{2m(k-1)+1} + O(p^{(2m-1)k}).$$

Proof. Take $\alpha\in T^2\mathbb{A}$; we show $c_\alpha(T^{-2m}) = p^{2m(k-1)+1} + O(p^{(2m-1)k})$ as $k\to\infty$. For $\deg\alpha\leq 2$, the Theorem follows from Corollary 3.10 and Lemma 3.13. We argue now by induction on D. Suppose $D>2$, and suppose for all $\beta\in T^2\mathbb{A}$, $\deg\beta < D$, we have $c_\beta(T^{-2m}) = p^{2m(k-1)+1} + O(p^{(2m-1)k})$. We know

$$\begin{aligned}
c_\alpha(T^{-2m}) = p^{2m(k-1)+1} &+ p\sum_{D\leq n<2m} p^{n(k-1)}(-1|p)^{nk}c_0(T^{2n-2m})\\
&-\sum_{D-1\leq n<2m} p^{n(k-1)}(-1|p)^{nk}c_0(T^{2n-2m})\\
&+\sum_{\substack{1\leq n\leq \frac{D}{2}-1\\ \deg\beta = D-2n}} p^{n(k-1)}(-1|p)^{nk}c_\beta(T^{2n-2m})p^{n-D}K(\alpha T^{-n},\beta T^n;\mathfrak{P}_\infty^{D-n}).
\end{aligned}$$

We know $c_0(T^{2n-2m}) = O(p^{(2m-2n)k})$, so $p^{n(k-1)}(-1|p)^{nk}c_0(T^{2n-2m}) = O(p^{(2m-n)k})$. Similarly, $p^{n(k-1)}(-1|p)^{nk}c_\beta(T^{2n-2m}) = O(p^{(2m-n)k})$ for $\beta\in T^2\mathbb{A}$ of degree $D-2n$. Now, every Kloosterman sum has p^{D-n} summands, each of modulus 1. Hence we certainly have

$$|p^{n-D}K(\alpha T^{-n},\beta T^n;\mathfrak{P}_\infty^{D-n})|\leq 1.$$

Also, the number of terms in each sum depends only on D and m; in particular, the number of terms in the above sums is independent of k. Thus

$$\begin{aligned}
c_\alpha(T^{-2m}) &= p^{2m(k-1)+1} + \sum_{D-1\leq n<2m} O(p^{(2m-n)k})\\
&\quad + \sum_{1\leq n\leq\frac{D}{2}-1} O(p^{(2m-n)k})\\
&= p^{2m(k-1)+1} + O(p^{(2m-1)k}).
\end{aligned}$$

□

[Need to add references.]

Department of Mathematics, The Colorado College, Colorado Springs, Colorado (zip?)

Department of Mathematics, University of Colorado, Boulder, Colorado 80309-0395?