ANALYSIS 1A: Lemmas to help prove Proposition 2.8

Lemma 0.1. *Suppose that $x, y \in \mathbb{R}$ and that for any $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$, we have $x \leq y + \varepsilon$. Then $x \leq y$.*

Proof. For the sake of contradiction, suppose that $x > y$. So $x - y > 0$, and hence

$$\varepsilon = \frac{x-y}{2} > 0.$$

Then

$$2(y + \varepsilon) = 2y + x - y = y + x,$$

and $y + x < 2x$ since $y < x$. Thus

$$2(y + \varepsilon) < 2x$$

and so $y + \varepsilon < x$. But this contradics our initial assumption that for any $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$, we have $x \leq y + \varepsilon$. □

Lemma 0.2. *Suppose that $x, y \in \mathbb{R}$ with $x, y > 0$. Also suppose that for any $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$, we have $x \leq (y + \varepsilon)^2$. Then $x \leq y^2$.*

Proof. For the sake of contradiction, suppose that $y^2 < x$. Using the Archimedan Property, choose $n \in \mathbb{N}$ so that $y < n$.

[Scratch work: We want to find some $\varepsilon > 0$ so that $(y + \varepsilon)^2 < x$, giving us a contradiction to our initial assumption. We know that

$$\begin{aligned}(y+\varepsilon)^2 &= y^2 + 2y\varepsilon + \varepsilon^2 \\ &= y^2 + \varepsilon(2y + \varepsilon) \\ &< y^2 + \varepsilon(2n + 1)\end{aligned}$$

PROVIDED we choose $\varepsilon \leq 1$. To get $y^2 + \varepsilon(2n + 1) < x$, we can take

$$\varepsilon < \frac{x - y^2}{2n + 1}.$$

Since we also want $\varepsilon < 1$, we choose $\varepsilon = \min((x - y^2)/(2(2n + 1)), 1/2)$.]

Choose

$$\varepsilon = \min\left(\frac{x - y^2}{2(2n+1)}, \frac{1}{2}\right).$$

Then

$$\begin{aligned}(y+\varepsilon)^2 &= y^2 + \varepsilon(2y + \varepsilon) \\ &< y^2 + \varepsilon(2n + 1) \\ &\leq y^2 + \frac{x - y^2}{2(2n+1)}(2n + 1) \\ &= y^2 + \frac{x - y^2}{2} \\ &< y^2 + x - y^2 \\ &= x.\end{aligned}$$

Thus $(y + \varepsilon)^2 < x$, contradicting our hypothesis that for any $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$, we have $x \leq (y+\varepsilon)^2$. Hence it cannot be possible that $y^2 < x$, which means that we must have $x \leq y^2$. $\square$