Analysis 1A: Reformulation of Proposition 4.6

Here we state and prove a stronger version of Proposition 4.6 given in the notes.

Proposition 0.1. *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers, and suppose $\alpha \in \mathbb{R}$. The following are equivalent.*

(1) For all $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ so that for all $n \in \mathbb{N}$ with $n \geq N$, we have $|a_n - \alpha| \leq \varepsilon$.

(2) Suppose that $K > 0$. For all $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ so that for all $n \in \mathbb{N}$ with $n \geq N$, we have $|a_n - \alpha| \leq K\varepsilon$.

(1) For all $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ so that for all $n \in \mathbb{N}$ with $n \geq N$, we have $|a_n - \alpha| < \varepsilon$.

Proof. We show that (3) $\implies$ (1), (1) $\implies$ (2), and (2) $\implies$ (3). This will show that each of these statements implies the other two; for instance, if we know that (3) $\implies$ (1) and (1) $\implies$ (2), and we then assume that (3) holds, then (1) must hold (as (3) $\implies$ (1)), and hence (2) must also hold (as (1) $\implies$ (2)).

To show that (3) $\implies$ (1): Suppose that (3) holds. Since $|a_n - \alpha| < \varepsilon$ implies that $|a_n - \alpha| \leq \varepsilon$, we have that (3) $\implies$ (1).

To show that (1) $\implies$ (2): Suppose that (1) holds; suppose that $K > 0$ and $\varepsilon' > 0$. Set $\varepsilon = K\varepsilon'$. So $\varepsilon > 0$, hence there exists some $N \in \mathbb{N}$ so that for all $n \in \mathbb{N}$ with $n \geq N$, we have $|a_n - \alpha| \leq \varepsilon = k\varepsilon'$. Thus with $K > 0$, for all $\varepsilon' > 0$, there exists some $N \in \mathbb{N}$ so that for all $n \in \mathbb{N}$ with $n \geq N$, we have $|a_n - \alpha| \leq K\varepsilon'$. As the only condition on ε' is that $\varepsilon' > 0$, we can now rename ε' as ε, to get statement (2).

To show that (2) $\implies$ (3): Suppose that (2) holds. Take $\varepsilon > 0$. Then with $K = 1/2$, (2) gives us that there exists some $N \in \mathbb{N}$ so that for all $n \in \mathbb{N}$ with $n \geq N$, we have $|a_n - \alpha| \leq K\varepsilon = \varepsilon/2 < \varepsilon$. Hence (3) holds. $\square$