A1 (exercises $4+4$ marks)

(a) We have that $0 \in A$ so $\sup A \geq 0$. On the otherhand if $x \in A$ then $x = 1-n$ for some $n \in \mathbb{N}$ and so $x \leq 0$. Therefore $\sup A = 0$. For the lower bound if $K \in \mathbb{R}$ then by the Archimedean principle there exists $n \in \mathbb{N}$ such that $n \geq |K|$. Thus $1-(n+1) \in A$ and $1-(n+1) \leq -|K| \leq K$. Thus A is unbounded below and so $\inf A = -\infty$.

(b) $|x-1| \geq |x-2|$ is equivalent to $x \geq 3/2$. Therefore

$$B = \{x \in \mathbb{R} : x \geq 3/2\} = [3/2, \infty).$$

This means $\sup B = \infty$ and $\inf B = 3/2$.

A2 (bookwork+standard 2+3+3 marks)

(a) We have that $\lim_{n\to\infty} a_n = \alpha$ if and only if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for any $n \in \mathbb{N}$ with $n \geq N$ we have that

$$|a_n - \alpha| \leq \epsilon.$$

(b) (i) We have that $\frac{n+7}{n^2+9} = \frac{n^{-1}+7n^{-2}}{1+9n^{-2}}$. We know that $\lim_{n\to\infty} n^{-2} = 0$ and $\lim_{n\to\infty} n^{-1} = 0$ and so by the arithmetic properties of limits $\lim_{n\to\infty} n^{-1} + 7n^{-1} = 0$, $\lim_{n\to\infty} 1 + 9n^{-2} = 1$ and

$$\lim_{n\to\infty} \frac{n+7}{n^2+9} = 0.$$

(ii) We have that for all $n \in \mathbb{N}$

$$b_n = \frac{2^n + 5}{2^n} = \frac{1 + 5 \cdot 2^{-n}}{1}.$$

We know that $\lim_{n\to\infty} 2^{-n} = 0$ and so by the arithmetic properties of limits of sequences

$$\lim_{n\to\infty} b_n = 1.$$

A3 (bookwork+exercises $3+5$ marks)

(a) A sequence (a_n) is a Cauchy sequence if and only if for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n, m \in \mathbb{N}$ and $n, m \geq N$ then $|a_n - a_m| \leq \epsilon$.

(b) We know that (b_n) is convergent so we can let $b \in \mathbb{R}$ satisfy $\lim_{n\to\infty} b_n = b$. Let $\epsilon > 0$ choose $N \in \mathbb{N}$ such that if $n \in \mathbb{N}$ and $n \geq N$ then $|b_n - b| \leq \epsilon/2$. We then have that if $n, m \in \mathbb{N}$ and $n, m \geq N$ then

$$|b_n - b_m| = |b_n - b + b - b_m| \leq |b_n - b| + |b - b_m| \leq \epsilon/2 + \epsilon/2 = \epsilon.$$

A4 (Exercises $2+3+3$ marks)

(a) In this case $\lim_{n\to\infty} \frac{n^2}{5n^2+1} = 1/5 \neq 0$ and so the series cannot be convergent.

(b) We have that

$$\frac{a_{n+1}}{a_n} = \frac{\frac{2^{n+1}}{n+1!}}{\frac{2^n}{n!}} = \frac{2}{n+1} \to 0 \text{ as } n \to \infty$$

and so by the ratio test the series is convergent.

(c) We have that $0 \leq \frac{2}{2n^2-1} \leq \frac{2}{2n^2-n^2} = \frac{2}{n^2}$ and since $\sum_{n=1}^{\infty} n^{-2}$ is convergent, so is $\sum_{n=1}^{\infty} \frac{2}{n^2}$ and we have that $\sum_{n=1}^{\infty} \frac{2}{2n^2-1}$ is convergent by the comparison test.

A5 (bookwork+exercises $3 + 5$ marks)

(a) Let $a, b \in \mathbb{R}$ with $a < b$. Let $f : [a, b] \to \mathbb{R}$ be a continuous function where $f(a) < f(b)$. For all $y \in (f(a), f(b))$ there exists $x \in (a, b)$ such that $f(x) = y$.

(b) Define $f : [0, 1] \to \mathbb{R}$ by $f(y) = y^3+y^2-7y+4$ for all $y \in [0, 1]$. f is a polynomial and so is continuous. We have that $f(0) = 4$ and $f(1) = -1$. Thus by the intermediate value theorem (applied to $-f$) we have that there exists $x \in (0, 1)$ such that $f(x) = 0$. Therefore $x^3 + x^2 + 4 = 7x$.

SECTION B

(1) (a) (bookwork 3 points, exercises $5 + 6$ points)

(i) Let $(a_n)_{n\in\mathbb{N}}$ be a real valued sequence. If (a_n) is monotone increasing and bounded above then (a_n) is convergent.

(A) Since $(1-2/n) \leq 1$ for all $n \in \mathbb{N}$ we have that since f is increasing $a_n = f(1-2/n) \leq f(1)$ so a_n is bounded above by $f(1)$. If $m \geq n$ then $2/m \leq 2/n$ and so $1 - 2/n \geq 1 - 2/m$. Therefore since f is increasing $a_n = f(1-2/n) \leq f(1-2/m) = a_m$. Therefore (a_n) is bounded above and monotone increasing and so by the Monotone convergence theorem (a_n) is convergent.

(B) We know that for all $x \in (0, 2]$, $f(x) \leq f(2)$. Thus if f is bounded above and since f is unbounded it must be unbounded below. Let $k \in \mathbb{R}$. Since f is unbounded below we can choose $\delta \in (0, 2]$ such that $f(\delta) \leq k$. Now if $x \in (0, 2]$ and $|x| \leq \delta$ then since f is increasing $f(x) \leq f(\delta) \leq K$. Therefore $\lim_{x\to 0} f(x) = -\infty$.

(b) (bookwork 3 marks) Let $a, b \in \mathbb{R}$ with $a < b$. If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ then f is bounded and there exist $x, y \in [a, b]$ where

$$f(x) = \sup\{f(c) : c \in [a, b]\}$$

and

$$f(y) = \inf\{f(c) : c \in [a, b]\}.$$

(c) (exercise 5 marks). Define $f : (1, 2) \to \mathbb{R}$ by $f(x) = \frac{1}{x-1}$. f is a rational function and so is continuous on $(1, 2)$. f is unbounded above since if we let $k \in (1, \infty)$ then $(1 + k)/k \in (1, 2)$ and $f((1 + k)/k) = k$.

(d) (unseen 8 marks). We define $h : [0, 1] \to \mathbb{R}$ by $h(x) = f(x)/g(x)$ for all $x \in [0, 1]$. This is well defined since $g(x) \neq 0$ for any $x \in [0, 1]$ and continuous on $[0, 1]$ since f and g are continuous on $[0, 1]$. We know that since $f(x) < g(x)$ for all $x \in [0, 1]$ that $h(x) < 1$ for all $x \in [0, 1]$. If we let $\alpha = \sup\{h(x) : x \in [0, 1]\}$ then by the extremal value theorem there exists $x \in [0, 1]$ such that $h(x) = \alpha$. We know that $\alpha < 1$ since $h(x) < 1$. Thus

$h(y) \leq h(x) = \alpha$ for all $y \in [0,1]$ and so $\frac{f(y)}{g(y)} \leq \alpha$ for all $y \in [0,1]$. Therefore since $g(y) > 0$ for all $y \in [0,1]$ we have that $f(y) \leq \alpha g(y)$ for all $y \in [0,1]$.

(2) (a) (bookwork 3 marks). The Bolzano-Weierstrass Theorem states that if (a_n) is a bounded sequence of real numbers then (a_n) contains a convergent subsequence.

(b) (exercise 5 marks) Take the sequence (a_n) where $a_n = n + (-1)^n n$ for all $n \in \mathbb{N}$. Then $a_{2n-1} = 0$ for all $n \in \mathbb{N}$ so (a_n) contains a convergent subsequence but $a_{2n} = 4n$ for all $n \in \mathbb{N}$ and so (a_n) is unbounded.

(c) (bookwork 5 marks) Let (a_{n_k}) be a subsequence of (a_n). Let $r \in \mathbb{R}$ and choose $K \in \mathbb{N}$ such that if $k \geq K$ then $|a_k| \geq r$. Now since (n_k) is a strictly monotone increasing sequence of natural numbers if $k \geq K$ then $n_k \geq k \geq K$. Thus for all $k \geq K$, $|a_{n_k}| \geq r$ and so $\lim_{k\to\infty} |a_{n_k}| = \infty$.

(d) (exercise 6 marks) If $\lim_{n\to\infty} |a_n| = \infty$ by the previous part we know that all subsequence a_{n_k} satisfy that $\lim_{k\to\infty} |a_{n_k}| = \infty$ and so no subsequences are convergent. So we need to show if $\lim_{n\to\infty} |a_n| = \infty$ is not true then (a_n) does contain a convergent subsequence. Since $\lim_{n\to\infty} |a_n| = \infty$ is not true there exists $r \in \mathbb{R}$ such that for all $N \in \mathbb{N}$ there exists $n \geq N$ where $|a_n| \leq r$. So we con construct a bounded subsequence inductively as follows. Choose $n_1 \in \mathbb{N}$ such that $|a_{n_1}| \leq r$. For $k \in \mathbb{N}$ given n_k we can find $n_{k+1} \geq n_k + 1$ such that $|a_{n_{k+1}}| \leq r$. This gives us a bounded subsequence (a_{n_k}). By the Bolzano-Weierstrass Theorem the subsequence (a_{n_k}) contains a convergent subsequenc which will be a subsequence of the original sequence.

(e) (bookwork 4 marks) Let $a_n = n^{-1}$ for all $n \in \mathbb{N}$. Then $\sum_{n=1}^{\infty} a_n$ is divegent but $\lim_{n\to\infty} a_n = 0$.

(f) (unseen 7 marks) We know that the series $\sum_{n=1}^{\infty} 2^{-n}$ is convergent (any convergent series with all positive terms will do). We define a subsequence inductively as follows. Choose n_1 such that $a_{n_1} \leq 2^{-1}$ (we can do this since $\lim_{n\to\infty} a_n = 0$). For $k \in \mathbb{N}$ given n_k in $\mathbb{N}$ we can choose n_{k+1} such that $a_{n_{k+1}} \leq 2^{-(k+1)}$ and $n_{k+1} > n_k$. Therefore $0 \leq a_{n_k} \leq 2^{-k}$. Therefore by the comparison test $\sum_{k=1}^{\infty} a_{n_k}$ is convergent.