Solutions to Exercise sheet 1: Rationals and irrationals

1. Let $x \in \mathbb{Q}$ with $x \neq 0$. Since $x \neq 0$ it has a multiplicative inverse $x^{-1} = 1/x \in \mathbb{Q}$ and if $xy \in Q$ then $x^{-1}xy \in \mathbb{Q}$ and thus $y \in \mathbb{Q}$. So it is not possible to have a nonzero $x \in \mathbb{Q}$ and $y \notin \mathbb{Q}$ where $xy \in \mathbb{Q}$.

2. To see this first of all note that $(1 + \sqrt{2}) - \sqrt{2} = 1 \in \mathbb{Q}$. We know that since $\sqrt{2} \notin \mathbb{Q}$ by the first question $-\sqrt{2} \notin \mathbb{Q}$ and if $1 + \sqrt{2} \in \mathbb{Q}$ then $1 + \sqrt{2} - 1 \in \mathbb{Q}$ which is clearly a contradiction. So $\sqrt{2} + 1 \notin \mathbb{Q}$ and we have found two irrationals x, y with $x + y \in \mathbb{Q}$ ($x = 1 + \sqrt{2}$, $y = -\sqrt{2}$.)

3. (a) The integers are not bounded by any rational number. Let $\frac{p}{q} \in \mathbb{Q}$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ then

$$\left|\frac{p}{q}\right| \leq q|\frac{p}{q}| = |p| < 2|p|.$$

So for any rational number x there exists an integer k with $k > x$ and so no rational number can bound the integers. (By now you will also have seen that no real numbers bound the integers either, this is the Archimedean principle).

(b) The set A_2 is bounded by 1. All elements of A_2 are of the form $\frac{n-1}{n}$ where $n \in \mathbb{N}$ and for all $n \in \mathbb{N}$,

$$\left|\frac{n-1}{n}\right| = \frac{n-1}{n} = 1 - \frac{1}{n} \leq 1.$$

4. (a) $A_1 = \left\{\frac{n+6}{6} : n \in \mathbb{N}\right\}$, and take $x \in \mathbb{Q}$. [We want to find some $n \in \mathbb{N}$ so that $\frac{n+5}{6} > x$. We have $\frac{n+5}{6} > x \iff n > 6x - 5$.] By the Archimedean Property, there is some $n \in \mathbb{N}$ with $n > 6x - 5$, and hence $\frac{n+5}{6} > x$. As this argument is valid for any $x \in \mathbb{Q}$, this shows that A_1 does not have an upper bound in $\mathbb{Q}$.

ALTERNATIVELY: We need to show that whichever $x \in \mathbb{Q}$ we are given we can find an element $a \in A_1$ with $a > x$. So fix $x = \frac{p}{q}$ where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. If $x \leq 0$ we can simply note that for $n = 1$, $(n + 5)/6 = 1$ and so $1 > x$ and $1 \in A_1$. If $x = p/q > 0$ (in fact it is ok to skip the first part) then we want to find $n \in \mathbb{N}$ where $(n + 5)/6 > p/q$ and if we use that $p/q \leq p$ then we need to find $n \in \mathbb{N}$ where

$$n + 5 > 6p.$$

To do this we can take $n = 6p + 1$, then $n \in \mathbb{N}$ and $(n + 5)/6 = (6p + 6)/6 > p$. So $(5n + 1)/6 \in A_1$ and is bigger than x. Thus A_1 is unbounded above in $\mathbb{Q}$.

(b) $A_2 = \left\{ \frac{n+6}{n} : n \in \mathbb{N} \right\}$. [To show that A_2 is bounded in $\mathbb{Q}$, we need to find $\alpha, \beta \in \mathbb{Q}$ so that α is a lower bound for A_2 and β is an upper bound for A_2. For $n \in \mathbb{N}$, we have $\frac{n+6}{n} = 1 + \frac{6}{n}$, which is always positive. When $n = 1$, $1 + \frac{6}{n} = 7$.] Since $\frac{n+6}{n} > 0$ for all $n \in \mathbb{N}$, 0 is a lower bound for A_2 [in fact, any rational r with $r \leq 1$ is a lower bound for A_2]. For $n \in \mathbb{N}$, we have $1 \leq n$, so $6 \leq 6n$; thus $n + 6 \leq 7n$ and hence $\frac{n+6}{n} \leq 7$. Thus 7 is an upper bound for A_2 [in fact, any rational s with $s \geq 7$ is an upper bound for A_2].

ALTERNATIVELY: We need to find $x \in Q$ such that for all $n \in \mathbb{N}$, $\left|\frac{n+6}{n}\right| \leq x$. We have that for any $n \in \mathbb{N}$

$$\left|\frac{n+6}{n}\right| = \frac{n+6}{n} = 1 + \frac{6}{n}.$$

[We can use the axioms to show $\frac{6}{n} \leq 6$ for any $n \in \mathbb{N}$: with $n \in \mathbb{N}$, we know that $1 \leq n$ and so $6 \cdot \frac{1}{n} \cdot 1 \leq 6 \cdot \frac{1}{n} \cdot n$, and so $\frac{6}{n} \leq 6$.] Since $\frac{6}{n} \leq 6$ for any $n \in \mathbb{N}$, we have

$$\left|\frac{n+6}{n}\right| \leq 1 + 6 = 7$$

for any $n \in \mathbb{N}$. So for all $a \in A_2$ we have that $|a| \leq 7$ and so A_2 is bounded in $\mathbb{Q}$.

5. Since A is bounded in $\mathbb{Q}$, there are $\alpha, \beta \in \mathbb{Q}$ so that α is a lower bound for A and β is an upper bound for A. This means that for all $x \in A$, we have $\alpha \leq x$ and $x \leq \beta$. Similarly, since B is bounded in $\mathbb{Q}$, there are $\gamma, \delta \in \mathbb{Q}$ so that for all $x \in B$, we have $\gamma \leq x$ and $x \leq \delta$. [Consider a special case: What if $\alpha = 3$ and $\gamma = 5$; could we use one of these as a lower bound for *both* A and B?] Set $\lambda = \min(\alpha, \gamma)$ [so λ is the smaller of the two numbers α and γ]. Thus $\lambda \leq \alpha$ and $\lambda \leq \gamma$. Set $\kappa = \max(\beta, \delta)$ [so κ is the larger of the two numbers β and δ]. Thus $\beta \leq \kappa$ and $\delta \leq \kappa$.

We claim that λ is a lower bound for $A \cup B$, and κ is an upper bound for $A \cup B$. Take $x \in A \cup B$. First suppose that $x \in A$. So we have $\lambda \leq \alpha \leq x \leq \beta \leq \kappa$, and hence $\lambda \leq x \leq \kappa$. Now suppose that $x \in B$. So we have $\lambda \leq \gamma \leq x \leq \delta \leq \kappa$, and hence $\lambda \leq x \leq \kappa$. Thus λ is a lower bound for $A \cup B$, and κ is an upper bound for $A \cup B$.

ALTERNATIVELY: Suppose that A is bounded by $x \in \mathbb{Q}$ (i.e for all $a \in A$, $|a| \leq x$) and B is bounded by $y \in \mathbb{Q}$. We let $z = \max\{x, y\}$ and show that $z \in \mathbb{Q}$ bounds $A \cup B$. Let $a \in A \cup B$ and note that this means $a \in A$ or $a \in B$. In the case when $a \in A$ we have that $|a| \leq x \leq z$. On the other hand if $a \in B$ then $|a| \leq y \leq z$.

6. We need to show (1) if $\sqrt{m} \in \mathbb{N}$ then $\sqrt{m} \in \mathbb{Q}$, and (2) if $\sqrt{m} \in \mathbb{Q}$ then $\sqrt{m} \in \mathbb{N}$.

   We first prove (1). Suppose that $\sqrt{m} \in \mathbb{N}$. Since $\mathbb{N} \subseteq \mathbb{Q}$, then $\sqrt{m} \in \mathbb{Q}$.

   So now we prove (2). Suppose that $\sqrt{m} \in \mathbb{Q}$. We want to conclude that $\sqrt{m} \in \mathbb{N}$, s for the sake of contradiction, suppose that $\sqrt{m} \notin \mathbb{N}$. [Following the proof of Theorem 2.3, with $\sqrt{2}$ replaced by $\sqrt{m}$, we get the following.] Since $\sqrt{m} \in \mathbb{Q}$, we have $\sqrt{m} = a/b$ for some $a \in \mathbb{Z}$ and $b \in \mathbb{N}$. Hence $b\sqrt{m} \in \mathbb{Z}$. Take q to be the smallest element of $\mathbb{N}$ so that $q\sqrt{m} \in \mathbb{Z}$. [Next, the proof of Theorem 2.3 uses that $1 < \sqrt{2} < 2$. As per the hint, we need to find $k \in \mathbb{N}$ so that $k < \sqrt{m} < k+1$. Then we continue to mimic the proof of Theorem 2.3 with $\sqrt{2}$ replaced by $\sqrt{m}$, and the inequality $1 < \sqrt{2} < 2$ replaced by $k < \sqrt{m} < k+1$.] Choose $k \in \mathbb{N}$ to be the largest element of $\mathbb{N}$ with $k < \sqrt{m}$. Since $k+1 \in \mathbb{N}$ with $k < k+1$, we must have $\sqrt{m} \leq k+1$. But we have assumed that $\sqrt{m} \notin \mathbb{N}$, so we have $\sqrt{m} < k+1$. Set $q' = q(\sqrt{m} - k)$. So $q' = q(\sqrt{m} - k) \in \mathbb{Z}$ (since $q\sqrt{m}, k \in \mathbb{Z}$). Since $0 < \sqrt{m} - k < 1$, we have $0 < q(\sqrt{m} - k) < q$ (so $0 < q' < q$). However,

   $$\sqrt{m}q' = \sqrt{m}(q(\sqrt{m} - k)) = mq - kq\sqrt{m} \in \mathbb{Z}$$

   since $mq, k, q\sqrt{m} \in \mathbb{Z}$. But we chose q to be the smallest element of $\mathbb{N}$ so that $q\sqrt{m} \in \mathbb{Z}$, and yet $q' \in \mathbb{N}$ with $q' < q$ and $q'\sqrt{m} \in \mathbb{Z}$. This is a contradiction! So when $m \in \mathbb{N}$ with $\sqrt{m} \in \mathbb{Q}$, it must be the case that $\sqrt{m} \in \mathbb{N}$.

7. We have that $(\sqrt{3} + \sqrt{5})^2 = 3 + 5 + 2\sqrt{15}$. We know that $\sqrt{15}$ is irrational by the previous question ($3 < \sqrt{15} < 4$). Now if we then suppose that $8 + 2\sqrt{15} \in \mathbb{Q}$ we would have that $\sqrt{15} = \frac{8+2\sqrt{15}-8}{2} \in \mathbb{Q}$ which is a contradiction. So $(\sqrt{3}+\sqrt{5})^2 \notin \mathbb{Q}$. However if $(\sqrt{3}+\sqrt{5}) \in \mathbb{Q}$ then since the rationals are closed under multiplication $(\sqrt{3}+\sqrt{5})^2 \in \mathbb{Q}$ so we can conclude that $\sqrt{3} + \sqrt{5} \notin \mathbb{Q}$.

8. Consider $x = a + \frac{b-a}{\sqrt{2}}$. Since $\sqrt{2} \notin \mathbb{Q}$ we have that $x \notin \mathbb{Q}$. We also have

   $$a < x = a + \frac{b-a}{\sqrt{2}} < a + b - a = b.$$