Solutions to Exercise sheet 2: Inequalities and induction

1. (a) For all $x \in \mathbb{R}$ we have that 0 is an additive identity (A3) and that

$$x \cdot 0 = x \cdot (0 + 0)$$

By A11 (the distributitivity law) we have $x \cdot (0+0) = x \cdot 0 + x \cdot 0$. Thus

$$x \cdot 0 = x \cdot 0 + x \cdot 0.$$

Using A4 (the existence of an additive inverse for all elements of $\mathbb{R}$) we can add $-(x \cdot 0)$ to each side to get

$$-(x \cdot 0) + x \cdot 0 = -(x \cdot 0) + x \cdot 0 + x \cdot 0.$$

Using that $(-x \cdot 0) + x \cdot 0 = 0$ gives us

$$0 = 0 + x \cdot 0 = x \cdot 0.$$

(b) We consider $y((-x) + x)$. By A4 (existence of additive inverse) and the previous exercise we have $y \cdot ((-x) + x) = y \cdot 0 = 0$. So by the distributive law (A11) and the fact that multiplication is commutative (A10), we have

$$0 = y \cdot ((-x) + x) = y \cdot (-x) + y \cdot x = (-x) \cdot y + x \cdot y.$$

Now adding $-(x \cdot y)$ (the additive inverse of $x \cdot y$) to both sides of the above equation, we get

$$0 + -(x \cdot y) = (-x) \cdot y + x \cdot y + -(x \cdot y).$$

Finally using the properties of the additive identity (A3), additive inverses (A4), and that addition is commutative (A5) we get

$$-(x \cdot y) = (-x) \cdot y.$$

(c) To do this we use part (a) and $A4$ to get $-x \cdot (y + (-y)) = -x \cdot 0 = 0$. On the other hand, by A11 we have $-x \cdot (y + (-y)) = -x \cdot y + (-x) \cdot (-y)$. So

$$0 = -x \cdot (y + (-y)) = -x \cdot y + (-x) \cdot (-y).$$

Now add $x \cdot y$ to both sides and use part (b) along with A4 and A3 to get

$$x \cdot y = x \cdot y + -x \cdot y + (-x \cdot)(-y) = 0 + (-x) \cdot (-y) = (-x) \cdot (-y).$$

(d) We start by supposing that $x < y$ and $z < 0$. We first show that $-z > 0$. To do this use O3 (compatibility with addition to get) $z + (-z) < -z$ and then A4 to get $0 < -z$. Now by compatibility with multiplication (O4) we get $x \cdot (-z) < y \cdot (-z)$. By part (b) and A10 this means $-x \cdot z < -y \cdot z$. Now use O3 to get

$$x \cdot z + y \cdot z - x \cdot z < x \cdot z + y \cdot z - y \cdot z.$$

Finally using A5, A4 and A3 we get $x \cdot z + y \cdot z - x \cdot z = y \cdot z + 0 = y \cdot z$ and $x \cdot z + y \cdot z - y \cdot z = x \cdot z + 0 = x \cdot z$. So $y \cdot z < x \cdot z$.

2. We can prove this using the Archimedean principle. Let $a, b > 0$ and consider $\frac{b}{a}$. By the Archimedean principle we can find $n \in \mathbb{N}$ with $n \geq \frac{b}{a}$ and thus $(n+1) \in \mathbb{N}$ with $(n+1) > \frac{b}{a}$. We then have $n+1 \in \mathbb{N}$ and $(n+1)a > b$.

3. We need to find a lower bound for the set A. So let $y \in A$ and note that this means $y = x^2 + 6x + 6$ for some $x \in \mathbb{R}$. Thus by completing the square and using that $z^2 \geq 0$ for any $z \in \mathbb{R}$ we have that

$$y = x^2 + 6x + 6 = (x+3)^2 - 3 \geq 0 - 3 = -3.$$

Thus A is bounded below by -3.

Let α denote the greatest lower bound for A. Since A is bounded below by -3, we have $-3 \leq \alpha$. Taking $x = -3$, we have

$$x^2 + 6x + 6 = (x-3)^2 - 3 = -3,$$

so $-3 \in A$. Thus $\alpha \leq -3$. Since we also have $-3 \leq \alpha$, we must have that $\alpha = -3$, meaning -3 is the greatest lower bound for A.

4. (a) Fix $x, y \in \mathbb{R}$. We will use that both $||x| - |y||$ and $|x - y|$ are nonnegative so it suffices to show that $||x| - |y||^2 \leq |x - y|^2$. We have

$$\begin{aligned} ||x| - |y||^2 &= (|x| - |y|)^2 = |x|^2 + |y|^2 - 2|x||y| \\ &= x^2 + y^2 - 2|xy| \leq x^2 + y^2 - 2xy = (x-y)^2 = |x-y|^2. \end{aligned}$$

(This can also be solved via a case by case analysis.)

(b) Fix $x, y, z \in \mathbb{R}$. We have $|x - y| = |x - z + z - y|$. By the triangle inequality we have that

$$|x - z + z - y| \leq |x - z| + |z - y|$$

and hence

$$|x - y| \leq |x - z| + |z - y|.$$

5. [Recall that for $y, \gamma \in \mathbb{R}$, $|y| < \gamma \iff -\gamma < y < \gamma$. Here $\iff$ means "is equivalent to", or (equivalently) "if and only if". Also recall that by Proposition 2.11, with $a, b \geq 0$ we have $a > b \iff a^2 > b^2$.]

(a)
$$|x-3| \leq 4 \iff -4 \leq x-3 \leq 4 \iff -1 \leq x \leq 7.$$
So $|x-3| \leq 4$ exactly when $-1 \leq x \leq 7$.

(b) *Solution 1:* [We use the definition of the absolute value to consider 2 cases for the value of $|x-2|$.]
<u>Case 1.</u> Suppose $x \geq 2$; so $|x-2| = x-2$. Then
$$\begin{aligned} |x-2| > 8 &\iff x-2 > 8 \\ &\iff x > 10. \end{aligned}$$
We have $x \geq 2$ and $x > 10$ when $x > 10$.
<u>Case 2.</u> Suppose $x < 2$; so $|x-2| = 2-x$. Then
$$\begin{aligned} |x-2| > 8 &\iff 2-x > 8 \\ &\iff x < -6. \end{aligned}$$
So $|x-2| > 8$ exactly when either $x > 10$ or $x < -6$.

Solution 2: [Here we use Proposition 2.11 and the fact that $yz > 0$ if and only if either (1) $y, z > 0$, or (2) $y, z < 0$.]
$$|x-2| > 8 \iff (x-2)^2 > 64 \iff x^2 - 4x - 60 > 0.$$
The roots of $x^2 - 4x - 60$ are 10 and -6, so $x^2 - 4x - 60 = (x-10)(x+6)$. We have $(x-10)(x+6) > 0$ exactly when either (1) $x > 10$ and $x > -6$, or (2) $x < 10$ and $x < -6$. So $|x-2| > 8$ exactly when either $x > 10$ or $x < -6$.

(c) *Solution 1:* We consider the cases that $x \geq 7$ and $x < 7$.
<u>Case 1.</u> Suppose that $x \geq 7$; so $|x-7| = x-7$ and $|x-5| = x-5$. So
$$|x-5| \leq |x-7| \iff x-5 \leq x-7 \iff -5 \leq -7.$$
Since it is not true that $-5 \leq -7$, it is not true that $|x-5| \leq |x-7|$ when $x \geq 7$.
<u>Case 2.</u> Suppose that $x < 7$. Then $|x-7| = 7-x$, and
$$\begin{aligned} |x-5| \leq |x-7| &\iff x-7 \leq x-5 \leq 7-x \\ &\iff 0 \leq 2 \leq 14-2x \\ &\iff 2 \leq 14-2x \\ &\iff 2x \leq 12 \\ &\iff x \leq 6. \end{aligned}$$

[Here we used that since $0 \leq 2$, we have the equivalence $0 \leq 2 \leq 14 - 2x \iff 2 \leq 14 - 2x$.]
So $|x-5| \leq |x-7|$ exactly when $x < 7$ and $x \leq 6$, or more simply, when $x \leq 6$.

Solution 2: [Here we use Propostion 2.11.]

$$\begin{aligned} |x-5| \leq |x-7| &\iff (x-5)^2 \leq (x-7)^2 \\ &\iff x^2 - 10x + 25 \leq x^2 - 14x + 49 \\ &\iff 4x \leq 24 \\ &\iff x \leq 6. \end{aligned}$$

Thinking geometrically: We have $|x-5| \leq |x-7|$ when the distance from x to 5 is less than or equal to the distance from x to 7, which is the case exactly when $x \leq 6$. [However, if you are trying to convince someone of this and that someone doesn't so easily see things geometrically, you would be well-advised to use one of the other solutions. Remember that when you are writing solutions to the homework, you should be aiming to write an argument that would convince all of your classmates.]

(d) Here $x \neq -1$. By Proposition 2.9, $\left|\frac{x-2}{x+1}\right| = |x-2|/|x+1|$, so

$$\left|\frac{x-2}{x+1}\right| \leq 2 \iff |x-2| \leq 2|x+1|.$$

Solution 1: We consider the cases $x > -1$ and $x < -1$.
<u>Case 1.</u> Suppose $x > -1$. So $|x+1| = x+1$. Then

$$\begin{aligned} |x-2| \leq 2|x+1| &\iff -2(x+1) \leq x-2 \leq 2(x+1) \\ &\iff 0 \leq 3x \leq 4x+4 \\ &\iff 0 \leq 3x \text{ and } 3x \leq 4x+4 \\ &\iff 0 \leq x \text{ and } -4 \leq x \\ &\iff 0 \leq x. \end{aligned}$$

<u>Case 2.</u> Suppose $x < -1$. So $|x+1| = -x-1$. Then

$$\begin{aligned} |x-2| \leq 2|x+1| &\iff 2(x+1) \leq x-2 \leq -2(x+1) \\ &\iff 4x+4 \leq 3x \leq 0 \\ &\iff 4x+4 \leq 3x \text{ and } 3x \leq 0 \\ &\iff x \leq -4 \text{ and } x \leq 0 \\ &\iff x \leq -4. \end{aligned}$$

So for $x \neq -1$, we have $\left|\frac{x-2}{x+1}\right| \leq 2$ exactly when we have either (1) $x > -1$ and $x \geq 0$, or (2) $x < -1$ and $x \leq -4$. Simplifying, we have $\left|\frac{x-2}{x+1}\right| \leq 2$ exactly when we have either (1) $x \geq 0$, or (2) $x \leq -4$.

Solution 2. [Here we use Proposition 2.11.]

$$\begin{aligned} |x-2| \leq 2|x+1| &\iff (x-2)^2 \leq \big(2(x_1)\big)^2 \\ &\iff x^2 - 4x + 4 \leq 4x^2 + 8x + 4 \\ &\iff 0 \leq 3x^2 + 12x \\ &\iff 0 \leq x(x+4). \end{aligned}$$

We have $0 \leq x(x+4)$ exactly when we have either (1) $x \geq 0$ and $x + 4 \geq 0$, or (2) $x \leq 0$ and $x + 4 \leq 0$. So we have $0 \leq x(x+4)$ exactly when we have either (1) $x \geq 0$ and $x \geq -4$ (i.e. $x \geq 0$), or (2) $x \leq 0$ and $x \leq -4$ (i.e. $x \leq -4$). Hence $\left|\frac{x-2}{x+1}\right| \leq 2$ exactly when we have either $x \geq 0$ or $x \leq -4$.

Thinking geometrically: We have $|x-2| \leq 2|x+1|$ exactly when the distance from x to 2 is less than or equal to twice the distance from x to -1, which is exactly when $x \geq 0$ or $x \leq -4$. [Again, if you are trying to convince someone of this and that someone doesn't so easily see things geometrically, you would be well-advised to use one of the other solutions.]

6. Fix $x > 0$. By the Binomial Theorem we have that for all $n \in \mathbb{N}$,

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k x^{n-k} = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}.$$

All of the terms in this summation are positive so we can write

$$(1+x)^n \geq \binom{n}{n-1} 1^{n-1} x^{n-(n-1)} + \binom{n}{n} 1^n x^{n-n} = nx + 1.$$

7. We let $P(n)$ be the statement that for all real numbers $x_1, \ldots, x_n$ we have

$$\left|\sum_{k=1}^{n} x_k\right| \leq \sum_{k=1}^{n} |x_k|.$$

We will prove this statement by induction. $P(1)$ reads that for all $x_1 \in \mathbb{R}$ we have that $|x_1| = |x_1$ and so is clearly true. We now suppose that for some $k \in \mathbb{N}$ the statement $P(k)$ is true. Now let $x_1, \ldots, x_k, x_{k+1} \in \mathbb{R}$ and let $y_k = x_1 + \cdots + x_k$. By the triangle inequality we have that

$$|x_1 + \ldots + x_k + x_{k+1}| = |y_k + x_{k+1}| \leq |y_k| + |x_{k+1}|.$$

By our inductive hypothesis $P(k)$ we have that

$$|y_k| = |x_1 + \cdots x_k| \leq |x_1| + \cdots + |x_k|.$$

Putting this together we have that

$$|X_! + \cdots x_k + x_{k+1}| \leq |x_1| + \cdots + |x_k| + |x_{k+1}|.$$

So we have shown that $P(k)$ implies $P(k+1)$ and so since $P(1)$ is true we can conclude that $P(k)$ is true for all $n \in \mathbb{N}$.

8. Let $P(n)$ be the statement that $n! \geq 2^n$.

   (Base case.) $P(4)$ says that $24 = 4! \geq 2^4 = 16$, which is true.

   (Induction step.) Suppose that $k \in \mathbb{N}$ with $k \geq 4$, and suppose that $P(k)$ is true. [So we are supposing that $k! \geq 2^k$; we want to show that $P(k+1)$ holds, or in other words, we want to show that $(k+1)! \geq 2^{k+1}$.] Using that $k! \geq 2^k$, we have

$$(k+1)! = (k+1)(k!) \geq (k+1)2^k.$$

   Since $k \geq 4$, we have $k+1 > 2$, so $(k+1)2^k > 2 \cdot 2^k = 2^{k+1}$. Hence we have

$$(k+1)! \geq (k+1)2^k > 2^{k+1},$$

   and so $(k+1)! \geq 2^{k+1}$. This shows that for $k \in \mathbb{N}$ with $k \geq 4$, if $P(k)$ is true then $P(k+1)$ is true. Since we already verified that $P(4)$ is true, the principe of mathematical induction tells us that $P(n)$ is true for all $n \in \mathbb{N}$ with $n \geq 4$.

9. Let $a, b \geq 0$. In this case $0 \leq (\sqrt{a} - \sqrt{b})^2 = a + b - 2\sqrt{a}\sqrt{b}$. So $\frac{a+b}{2} \geq \sqrt{a}\sqrt{b}$.

10. (a) If we suppose that $a_1 = a_n$ then $a_1 = a_2 = \cdots = a_n$. In this case

$$\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^n = a_1^n = \left(\prod_{k=1}^{n} a_k\right).$$

   (b) Suppose that $a_1 < a_n$; then

$$na_1 < (n-1)a_1 + a_n \leq \sum_{k=1}^{n} a_k \leq a_1 + (n-1)a_n < na_n.$$

   Dividing this chain of inequalities by n gives $a_1 < A_n < a_n$.

   (c) We have that

$$A_n(a_1+a_n-A_n)-a_1A_n = A_na_1+A_na_n-A_n^2-a_1a_n = (a_1-A_n)(A_n-a_n).$$

   By (b) we know that $a_1 - A_n < 0$ and $A_n - a_n < 0$ so $(a_1 - A_n)(A_n - a_n) > 0$.

(d) For $n \in \mathbb{N}$ we will let $P(n)$ be the statement that for any n and non-negative real numbers $a_1, \ldots, a_n$, we have that

$$\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^n \geq \left(\prod_{k=1}^{n} a_k\right).$$

It is immediate that $P(1)$ is true.

We now suppose that $P(m)$ is true for some $m \in \mathbb{N}$ with $m \geq 1$; we wish to show that this implies that $P(m+1)$ is true. Suppose that we have $m+1$ non-negative reals $a_1, \ldots, a_{m+1}$; without loss of generality, we may assume that $a_1 \leq a_2 \leq \cdots \leq a_{m+1}$. We know that if $a_1 = a_{m+1}$ the result is true by (a), so we may assume that $a_1 < a_{m+1}$. Now, we know that $a_{m+1} > A_{m+1}$ and so $a_1 + a_{m+1} - A_{m+1} \geq 0$. With $b_1 = a_1 + a_{m+1} - A_{m+1}$, by our induction hypothesis[i.e. the hypothesis that $P(m)$ is true], we have

$$\begin{aligned}\left(\frac{1}{m}\left(b_1 + a_2 + a_3 + \cdots + a_m\right)\right)^m &\geq b_1 a_2 \cdots a_m \\ &= (a_1 + a_{m+1} - A_{m+1}) a_2 \cdots a_m.\end{aligned}$$

As

$$\begin{aligned}b_1 + a_2 + \cdots + a_m &= (a_1 + a_2 + \cdots + a_m + a_{m+1}) - A_{m+1} \\ &= (m+1)A_{m+1} - A_{m+1},\end{aligned}$$

we have

$$\begin{aligned}A_{m+1}^m &= \left(\frac{1}{m}(b_1 + a_2 + \cdots + a_{m+1})\right)^m \\ &\geq (a_1 + a_{m+1} - A_{m+1}) a_2 \cdots a_m.\end{aligned}$$

By (c), we have $A_{m+1}(a_1 + a_{m+1} - A_{m+1}) - a_1 a_{m+1} > 0$ and we know $A_{m+1} > 0$, so

$$a_1 + a_{m+1} - A_{m+1} > \frac{a_1 a_{m+1}}{A_{m+1}}.$$

Thus we get

$$A_{m+1}^m \geq (a_1 + a_{m+1} - A_{m+1}) a_2 \cdots a_m > \frac{a_1 a_2 \cdots a_m a_{m+1}}{A_{m+1}}.$$

From this we get

$$A_{m+1}^{m+1} > a_1 \cdots a_{m+1},$$

showing that if $P(m)$ is true then $P(m+1)$ is true.

Hence by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.