Solutions to exercise sheet 3: Supremum and infimum

[Statements in square brackets are not necessary for a complete proof.]

1. (a) Let $x, y \in \mathbb{R}$ and suppose that for all $\varepsilon \in (0, \infty)$ we have that $x - y < \varepsilon$. We want to show that $x \leq y$. We proof this by contradiction. Suppose that $x > y$. In that case $x - y \in (0, \infty)$ and so $x - y < x - y$ which is clearly a contradiction. So $x \leq y$.

   (b) We have that for all $\varepsilon > 0$, $|x-y| < \varepsilon$ and thus $-\varepsilon < x-y < \varepsilon$. So by the first part of the question $x \leq y$. Moreover by multiplying through by -1 we can see that for all $\varepsilon > 0$ we have $y - x < \varepsilon$ and so by the first part of the question $y \leq x$. Since $y \leq x$ and $x \leq y$ we must have $x = y$.

2. Let $A, B \subseteq \mathbb{R}$ be non-empty and bounded with $A \subseteq B$. As A and B are bounded, this means that they have upper and lower bounds in $\mathbb{R}$, and hence $\sup A, \sup B, \inf A, \inf B \in \mathbb{R}$. [To show that $\sup A \leq \sup B$, we show that $\sup B$ is and upper bound for A, and hence $\sup A$ cannot exceed $\sup B$.] We take $a \in A$; so $a \in B$ [since $A \subseteq B$]. Since $a \in B$ and B is bounded above by $\sup B$, we have that $a \leq \sup B$. Thus $\sup B$ is an upper bound for A, which means that $\sup A \leq \sup B$.

   To show that $\inf A \geq \inf B$ is almost identical. Take $a \in A$ and note this means that $a \in B$. Since $a \in B$ and B is bounded below by $\inf B$, we know that $\inf B$ is a lower bound for A. Thus $\inf A \geq \inf B$.

3. Let $A \subseteq \mathbb{R}$ be bounded and non-empty and $B = \{|a| : a \in A\}$. We first consider the case when $|\sup A| \geq |\inf A|$. In this case $\sup B = \sup A$. Note that in this case we must have that since $\sup A \geq \inf A$, $\sup A \geq 0$. Let $b \in B$ then $b = |a|$ for some $a \in A$, if $b = a$ then $b \leq \sup A$. On the otherhand if $b = -a$ then $a \leq 0$ and $\inf A \leq$. We have that $a \geq \inf A$ and so $-a \geq -\inf A = |\inf A| \leq \sup A$. So $\sup A$ is an upper bound for B.

   Now let $y < \sup A$ (we need to show y cannot be an upper bound for B). We know that there exists $a \in A$ such that $a > y$ and so $|a| \geq a > y$ and since $|a| \in B$ y cannot be an upperbound for B. Therefore when $|\sup A| \geq |\inf A|$ we have that $\sup B = \sup A$.

   We now consider the case when $|\sup A| < |\inf A|$, in this case $\sup B = -\inf A$. This means that $\inf A < 0$. So if we let $b \in B$ then $b = |a|$ for some $a \in A$. If $b = a$ then $b \leq \sup A \leq |\inf A| = -\inf A$. If $b = -a$ then $a \geq \inf A$ and so $b \leq -\inf A$.

   Now let $y < -\inf A$ (we need to show y cannot be an upper bound for B). Thus we have that $-y > \inf A$ and so there exists $a \in A$ with $a < -y$ and thus $-a > y$. Thus $|a| \in B$ and $|a| \geq -a > y$ and so y

cannot be an upper bound for B. Therefore when $|\sup A| < |\inf A|$ we have that $\sup B = -\inf A$.

4. (a) [Recall: We know that $\inf(1, 3) = 1$ and $\sup(1, 3) = 3$.] For any $x \in A_1$ we have $1 < x < 3$ with $x \neq 2$. So 1 is a lower bound for A_1 and 3 is an upper bound for A_1. Take y so that $1 < y < 3$ [Scratch work: We want to show that y is not an upper bound or a lower bound for A_1. So we want some $a \neq 2$ with $y < a < 3$ and $b \neq 2$ with $1 < b < y$. We could try $a = \frac{3+y}{2}$ and $b = \frac{1+y}{2}$. But could a or b be equal to 2, and thus not in A_1? If $a = 2$ then $y = 1$, and if $b = 2$ then $y = 3$. So if $a = 2$ or $b = 2$ then we do not have $1 < y < 3$. Also,

$$a > y \iff 3 + y > 2y \iff 3 > y,$$

and $b < y \iff 1 + y < 2y \iff 1 < y$.] Set $a = \frac{3+y}{2}$ and $b = \frac{1+y}{2}$. So $1 < a < 3$, and $a \neq 2$ since $\frac{3+y}{2} = 2$ only if $y = 1$. Similarly, $b \neq 2$ since $\frac{1+y}{2} = 2$ only if $y = 3$. Hence $a, b \in A_1$. However, we claim that $b < y < a$, and so y is neither a lower bound nor an upper bound for A_1. To see this, note that since $3 > y$, we have $3 + y > 2y$ and so $a = \frac{3+y}{2} > y$; similarly, since $1 < y$ we have $1 + y < 2y$ and so $b = \frac{1+y}{2} < y$. Thus we must have $\inf A_1 = 1$ and $\sup A_1 = 3$.

(b) For any $n \in N$ we have that

$$0 \leq \frac{n-1}{n} = 1 - \frac{1}{n} \leq 1.$$

[Hence 0 is a lower bound for A_2 and 1 is an upper bound for A_2; recall that when a set A is bounded, $\inf A$ is the greatest lower bound for A and $\sup A$ is the least upper bound for A.] So $\sup A_2 \leq 1$ and $\inf A_2 \geq 0$. If we take $n = 1$ then we can see that $0 \in A_2$ and so $0 \geq \inf A_2$. Thus $\inf A_2 = 0$.

Now take $0 < y < 1$; we want to show that y is not an upper bound for A_2 (allowing us to conclude that 1 is the least upper bound for A_2). [Scratch work: We want $n \in \mathbb{N}$ so that $1 - \frac{1}{n} = \frac{n-1}{n} > y$. Since $1 - y > 0$, we have

$$1 - \frac{1}{n} > y \iff 1 - y > \frac{1}{n} \iff \frac{1}{1-y} < n.$$

So we can use the Archimedean Principle to produce this n.] As $0 < y < 1$, we know that $1 - y$, $\frac{1}{1-y}$ are positive real numbers. By the Archimedean Principle, there is some $n \in \mathbb{N}$ so that $\frac{1}{1-y} < n$ [we know there is some $k \in \mathbb{N}$ so that $\frac{1}{1-y} < k$, and we can take $n = k+1$]. Then [presenting our scratch work in reverse order] we have $1 - y > \frac{1}{n}$, and so $\frac{n-1}{n} = 1 - \frac{1}{n} > y$. Hence for $0 < y < 1$, y is not an upper bound for A_2. [If $y \leq 0$ then y is not an upper bound for A_2 since $\frac{2-1}{2} = \frac{1}{2} \in A_2$.] Thus we must have $\sup A_2 = 1$.

(c) [Scratch work: We must show that A_3 is not bounded below and is not bounded above. So for any $x \in \mathbb{R}$, we must find an element $a \in A_3$ so that $a < x$ and an element $b \in A_3$ so that $b > x$.] We fix $x \in \mathbb{R}$. For $n \in \mathbb{N}$ then $\frac{n^2+1}{n-\frac{1}{2}} \geq \frac{n^2}{n} = n$. By the Archimedean Principle we can find $n \in \mathbb{N}$ with $n > x$ and then

$$\frac{n^2+1}{n-\frac{1}{2}} \geq n > x.$$

Thus we have shown that A_3 is not bounded above and so $\sup A_3 = \infty$.

On the other hand for $n \in \mathbb{N}$ then

$$\frac{(-n)^2+1}{-n-\frac{1}{2}} = -\frac{n^2+1}{n+\frac{1}{2}} \leq -\frac{n^2}{2n} = -\frac{n}{2}.$$

So if we fix $x \in \mathbb{R}$ and by the Archimedean Principle we can choose $n \in \mathbb{N}$ such that $n > 2x$; then we have that

$$\frac{(-n)^2+1}{-n-\frac{1}{2}} \leq -\frac{n}{2} < x.$$

This means that for all $x \in \mathbb{R}$ we can find $a \in A_3$ with $a < x$ and so A_3 is not bounded below and so $\inf A_3 = -\infty$.

(d) [Scratch work: For $x \in (0, \infty)$, if $x > 1$ then $0 < \frac{x-1}{x} = 1 - \frac{1}{x}$, and if $x \leq 1$ then $\frac{x-1}{x} \leq 0$. So we will always have $\frac{x-1}{x} \leq 1$ for $x \in (0, \infty)$, and our experience with numbers tells us that for x really big, $\frac{1}{x}$ is really small. So it is reasonable to guess that $\sup A_4 = 1$. To get an idea of what $\inf A_4$ is, let's consider values of x between 0 and 1; for instance, consider $x = 1/2, 1/3, 1/4, 1/5, \ldots$. These values of x give us $1 - \frac{1}{x} = -1, -2, -3, -4, \ldots$. So it is reasonable to guess that $\inf A_4 = -\infty$, which means we need to show that A_4 is not bounded below.] We first show that $\sup A_4 = 1$. For any $x \in (0, \infty)$, we have $\frac{x-1}{x} = 1 - \frac{1}{x} < 1$ [we know that $\frac{1}{x} > 0$ since $x > 0$]. So A_4 is bounded above by 1. With $x = 2$, we have $\frac{x-1}{x} = \frac{1}{2} \in A_4$, so we know that any upper bound for A_4 must be positive [and in fact greater than or equal to $\frac{1}{2}$]. So take $y \in \mathbb{R}$ with $0 < y < 1$. [Scratch work: We want to find $x \in (0, \infty)$ so that $y < \frac{x-1}{x} = 1 - \frac{1}{x}$. So we want $\frac{1}{x} < 1 - y$. Since $\frac{1}{x}, 1 - y > 0$, we have $\frac{1}{x} < 1 - y$ exactly when $x > \frac{1}{1-y}$.] Using the Archimedean

Principle, we can choose $n \in \mathbb{N}$ with $n > \frac{1}{1-y}$. Since $1 - y > 0$, this means that $\frac{1}{n} < 1 - y$, and so $y < 1 - \frac{1}{n} = \frac{n-1}{n}$. Since $n \in (0, \infty)$, we have $\frac{n-1}{n} \in A_4$, and so y is not an upper bound for A_4. Thus 1 is the least upper bound for A_4, meaning that $\sup A_4 = 1$.

Now we show that $\inf A_4 = -\infty$. Choose $y \in \mathbb{R}$. [Scratch work: We want to find $x \in (0, \infty)$ so that $y > \frac{x-1}{x}$. In our previous scratch work, we saw that by choosing $x = 1/n$ with $n \in \mathbb{N}$, we get $\frac{x-1}{x} = 1 - n$. So we want to find $n \in \mathbb{N}$ with $1 - n < y$, or equivalently, $1 - y < n$.] By the Archimedean Principle, we can find $n \in \mathbb{N}$ so that $1 - y < n$, and hence $1 - \frac{1}{1/n} = 1 - n < y$. Taking $x = 1/n$, we get $1 - n = \frac{x-1}{x} \in A_4$ with $\frac{x-1}{x} < y$. Hence A_4 is not bounded below, which means that $\inf A_4 = -\infty$.

(e) [Scratch work: We can list some values of the set A_5 to get a feeling for the numbers in A_5. For $n \in \mathbb{N}$ with $n \leq 8$, we get $2, 7/3, 9/5, 0, 11/7, -1/4, 13/9 \in A_5$. Taking $n = 101$, we get $-96/102 \in A_5$. Except for $7/3$, the numbers in this list are bounded by 2, and $7/3 > 2$. So let's try to show that $7/3 = \sup A_5$.] Taking $n = 2$, we have $\frac{(-1)^n n+5}{n+1} = 7/3 \in A_5$. We claim that $7/3 = \sup A_5$. To show this, choose $n \in \mathbb{N}$. We have $\frac{(-1)^n n+5}{n+1} \leq \frac{n+5}{n+1}$. [Scratch work: For $n \in \mathbb{N}$, we have $\frac{n+1}{n+1} \leq \frac{7}{3} \iff 3(n+5) \leq 7(n+1) \iff 8 \leq 4n \iff 2 \leq n$.] When $n = 1$, we have $\frac{(-1)^n n+5}{n+1} = 2 < 7/3$. For $n \in \mathbb{N}$ with $n \geq 2$, we have $3(n+5) \leq 7(n+1)$, and hence $\frac{n+5}{n+1} \leq 7/3$; thus for $n \in \mathbb{N}$ with $n \geq 2$, we have $\frac{(-1)^n n+5}{n+1} \leq \frac{n+5}{n+1} \leq 7/3$. So $7/3$ is an upper bound for A_5, and since $7/3 \in A_5$, $7/3$ must the the least upper bound for A_5. So $\sup A_5 = 7/3$.

[Scratch work: To find $\inf A_5$, let's consider those $n \in \mathbb{N}$ with n odd, as these values of n sometimes correspond to elements of A_5 that are negative. So consider $n = 2m - 1$ where $m \in \mathbb{N}$. Then

$$\frac{(-1)^{2m-1}(2m-1)+5}{(2m-1)+1} = \frac{-2m+6}{2m} = -1 + \frac{3}{m}.$$

As m gets really large, $1/m$ gets really small. So it seems reasonable to guess that -1 is the greatest lower bound for A_5. Let's first ague that -1 is a lower bound for A_5.] Take $n \in \mathbb{N}$. When n is even we have $0 < \frac{(-1)^n n+5}{n+1}$. When n is odd, then $n = 2m-1$ for some $m \in \mathbb{N}$, and then $\frac{(-1)^n n+5}{n+1} = -1+3/m > -1$. Hence -1 is a lower bound for A_5. [Note: With $n = 7$ we have $\frac{(-1)^n n+5}{n+1} = -1/4 \in A_5$, so we know that any lower bound for A_5 must be negative.] Take y with $-1 < y$. [Scratch work: We want to find $m \in \mathbb{N}$ so that $-1 + 3/m < y$. So noting that $y + 1 > 0$, we want to find $m \in \mathbb{N}$ so that $m > \frac{3}{y+1}$.] Since $y + 1 > 0$, by the Archimedean Principle we can find $m \in \mathbb{N}$ so that $\frac{3}{y+1} < m$, and hence $y + 1 > \frac{3}{m}$. So

$$y > -1 + \frac{3}{m} = \frac{-2m + 6}{2m} = \frac{-(2m - 1) + 5}{(2m - 1) + 1}.$$

Taking $n = 2m - 1$, we have $n \in \mathbb{N}$ and $\frac{(-1)^n n+5}{n+1} \in A_5$ with $\frac{(-1)^n n+5}{n+1} < y$. So y is not a lower bound for A_5, which means that -1 is the greatest lower bound for A_5, i.e. $\inf A_5 = -1$.

5. (a) Suppose that $\sup A = \infty$. This means that for all $x \in \mathbb{R}$ we can find $a \in A$ such that $a > x$. To show that $\sup A - B = \infty$ we need to show for all $x \in \mathbb{R}$ there exists $z \in A - B$ such that $z > x$. To do this we fix $x \in \mathbb{R}$ and $b \in B$ (note we can do this since B is non-empty). Since $\sup A = \infty$ we can find $a \in A$ where $a > x + b$. Thus $a - b \in A - B$ and $a - b > x$. So $\sup A - B = \infty$.

   (b) Now suppose that both A and B are bounded and non-empty and so both have supremum and infimum in the real numbers. If we let $z \in A - B$ then we can find $a \in A$ and $b \in B$ where $z = a - b$. We then have that $a \leq \sup A$ and $b \geq \inf B$. So

$$z = a - b \leq \sup A - \inf B.$$

   Thus $\sup(A - B) \leq \sup A - \inf B$. Now we need to show that if $y < \sup A - \inf B$ then we can find $z > y$ with $z \in A - B$. To do this let $y = \sup A - \inf B - \varepsilon$ where $\varepsilon > 0$. We can find $a \in A$ with $a > \sup A - \varepsilon/2$ and $b \in B$ with $b < \inf B + \varepsilon/2$. Thus $a - b \in (A - B)$ and

$$(a - b) > \sup A - \varepsilon/2 - (\sup B - \varepsilon/2) = \sup A - \inf B - \varepsilon = y.$$

6. Firstly suppose that $\inf A = 0$. We wish to show $\sup B = \infty$. We let $x \in \mathbb{R}$. [We want to find $b \in B$ with $b > x$.] If $x \leq 0$ this is easy since we just take $a \in A$ (A is non-empty) and observe that $a^{-1} \in B$ with $a^{-1} > 0$. If $x > 0$ then since $\inf A = 0$ we can find $a \in A$ with $a < x^{-1}$ and so $ax < 1$ and thus $a^{-1} > x$ and since $a^{-1} \in B$ we have found $b \in B$ with $b > x$.

   Now suppose that $\inf A \neq 0$. Since $A \subseteq (0, \infty)$ and is non-empty this means that $\inf A \in (0, \infty)$. Now let $b \in B$. By the definition of B we can find $a \in A$ with $b = a^{-1}$. Since $a \geq \inf A$ we have that $a(\inf A)^{-1} \geq 1$ and so $b = a^{-1} \leq (\inf A)^{-1}$. Thus $\sup B \leq (\inf A)^{-1}$. Now let $y < (\inf A)^{-1}$. We can conclude that $y^{-1} > \inf A$ and so we can find $a \in A$ with $a < y^{-1}$ and thus $a^{-1} > y$. Since $a^{-1} \in B$ we can conclude that $\sup B \geq y$ and so $\sup B = (\inf A)^{-1}$.

7. (a) Let $x, y \in \mathbb{R}$ we have that $x^2 + xy + y^2 = (x + y/2)^2 + \frac{3y^2}{4} \geq 0$. Moreover when $x = y = 0$ we have $x^2 + xy + y^2 = 0$ and to have $(x+y/2)^2 + 3/4y^2 = 0$ we need that $y^2 = 0$ and $x = -y/2$ which gives that $x = y = 0$.

(b) If $x, y \in \mathbb{R}$ and $x^3 < y^3$ then $x^3 - y^3 < 0$ and so by factorising we have

$$(x - y)(x^2 + xy + y^2) < 0.$$

By the first part we know that $x^2+xy+y^2 > 0$ and so multiplying both sides by $(x^2 + xy + y^2)^{-1}$ yields that $x - y > 0$ and thus $x < y$.

(c) If we let $a \in A$ then $a^3 < 2 < y^3$ and so since $a^3 < y^3$ by the previous part of the question we must have that $a < y$.

(d) Let $y \in \mathbb{R}$ with $y^3 > 2$ and let $1 > \varepsilon > 0$. We have that

$$(y - \varepsilon)^3 = y^3 - 3\varepsilon y^2 + 3y\varepsilon^2 - \varepsilon^3.$$

Since $y > 0$ and $\varepsilon < 1$ we have

$$(y - \varepsilon)^3 \geq y^3 - 3\varepsilon y^2 - \varepsilon = y^3 - \varepsilon(3y^2 + 1).$$

We want to find $\varepsilon > 0$ where $(y - \varepsilon) > 2$ and so to do this we take $\varepsilon < \frac{y^3-2}{3y^2+1}$. This gives that

$$(y - \varepsilon)^3 > y^3 - (3y^2 + 1)\frac{y^3 - 2}{3y^2 + 1} = y^3 - y^3 + 2 = 2.$$

Since $(y - \varepsilon) > 2$ we know that for all $a \in A$ $a \leq (y - \varepsilon)$ and so $(y - \varepsilon)$ is an upper bound for A and thus $y > \sup A$.

(e) Now suppose that $y \in \mathbb{R}$ with $y^3 < 2$. We want to find $z \in \mathbb{R}$ with $z > y$ and $z^3 < 2$ in which case we know that $z \in A$ and so $\sup A > y$. Again we want to find ε sufficient small so that $(y + \varepsilon)^3 < 2$. However it helps to first deal with the case when $y < 1$. In this case since $1 \in A$ we know that $y \leq \sup A$. So now let $1 \leq y$ with $y^3 < 2$.
For $1 > \varepsilon > 0$ we have that

$$(y + \varepsilon)^3 = y^3 + 3y^2\varepsilon + 3y\varepsilon^2 + \varepsilon^3 \leq y^3 + 7\varepsilon.$$

So if we take $0 < \varepsilon < \frac{2-y^3}{7}$ we have that

$$(y + \varepsilon)^3 \leq y^3 + 7\frac{2 - y^3}{7} = 2.$$

(f) By the previous two parts we cannot have $(\sup A)^3 > 2$ or $\sup A < 2$ and so we must have that $(\sup A)^3 = 2$.

8. (a) For $A_1 = (0, 1]$ we know that $\sup A_1 = 1$ and $1 \in A_1$. So A_1 has a maximum value.

(b) For $A_2 = \left\{\frac{n-1}{n} : n \in \mathbb{N}\right\}$ we know by question 4 (b) that $\sup A_2 = 1$. However for all $n \in \mathbb{N}$

$$\frac{n-1}{n} = 1 - \frac{1}{n} < 1$$

and so $1 \notin A_2$ and we can conclude that A_2 does not have a maximum value.

(c) For $A_3 = \left\{\frac{1}{|n+\frac{1}{2}|} : n \in \mathbb{Z}\right\}$ we know that if $x \in A_3$ then $x = \frac{1}{|n+\frac{1}{2}|}$ for some $n \in \mathbb{Z}$ and since $|n + \frac{1}{2}| \geq \frac{1}{2}|$ for all $n \in \mathbb{Z}$ we have

$$\frac{1}{|n+\frac{1}{2}|} \leq 2.$$

So $\sup A_3 \leq 2$ but if we take $n = 0$ we have that $\frac{1}{|0+\frac{1}{2}|} = 2 \in A_3$ and so $\sup A_3 = 2$ and A_3 does have a maximum value.