Solutions to exercise sheet 4: Sequences and limits

1. (a) We have that for all $n \in \mathbb{N}$,

$$\left|\frac{n+3}{n}\right| = 1 + \frac{3}{n} \leq 1 + 3 = 4.$$

So this sequence is bounded.

(b) We have that for all $n \in \mathbb{N}$,

$$\left|\frac{n}{n+2}\right| = \frac{n}{n+2} \leq \frac{n}{n} = 1.$$

So this sequence is bounded.

(c) For $n \in \mathbb{N}$ we can write

$$\frac{n^2+1}{n+3} = \frac{1+n^-2}{n^-1+3n^{-2}} \geq \frac{1}{4n^{-1}} = \frac{n}{4}.$$

Thus if we let $x \in \mathbb{R}$ by the Archimedean principle we can find $n \in \mathbb{N}$ with $n > 4x$. Then

$$a_n = \frac{n^2+1}{n+3} \geq \frac{n}{4} > x.$$

Thus the sequence is unbounded.

(d) If we let $x \in \mathbb{R}$ by the Archimedean principle we can choose $n \in \mathbb{N}$ such that $n > x$. Thus $|a_n| = |-n| = n > x$. So the sequence is unbounded.

2. Let $\varepsilon > 0$. By the Archimedean principle we can choose $N \in \mathbb{N}$ with $N > \varepsilon^{-1}$ (**You may need to do some rough work before you figure this out**). Thus for $n \in \mathbb{N}$ with $n \geq N$ we have that

$$|a_n - 0| = \frac{1}{n+4} \leq \frac{1}{n} \leq \frac{1}{N} \leq \frac{1}{\varepsilon^{-1}} = \varepsilon.$$

3. Let $\varepsilon > 0$. By the Archimedean principle we can choose $N \in \mathbb{N}$ with $N > 2\varepsilon^{-1}$ (**You may need to do some rough work before you figure this out**). Thus for $n \in \mathbb{N}$ with $n \geq N$

$$|a_n - 1| = \left|\frac{n}{n+2} - 1\right| = \left|\frac{n-(n+2)}{n+2}\right| = \frac{2}{n+2} \leq \frac{2}{n} \leq \frac{2}{N} \leq \varepsilon.$$

4. (a) Let $\alpha \in \mathbb{R}$. We take $\varepsilon = 1$ and let $N \in \mathbb{N}$. We will use that $|a_N - a_{N+1}| = |2(-1)^N - 2(-1)^{N+1}| = 4$. Therefore by the triangle inequality

$$4 = |a_N - a_{N+1}| \leq |a_N - \alpha| + |a_{N+1} - \alpha|.$$

Thus we must have that either $|a_N - \alpha| > 1 = \varepsilon$ or $|a_{N+1} - \alpha| > 1$. Therefore we have found $\varepsilon > 0$ such that for all $N \in \mathbb{N}$ there exists $n \geq N$ (in this case either N or $N+1$) with $|a_n - \alpha| > \varepsilon$. Therefore we cannot have $\lim_{n\to\infty} a_n = \alpha$. Since this holds for all $\alpha \in \mathbb{R}$ we must have that (a_n) is a divergent sequence.

(b) Let $\alpha \in \mathbb{R}$. We take $\varepsilon = 1$ and let $N \in \mathbb{N}$. We will use that $|a_N - a_{N+1}| = |6(N) - 6(N+1)| = 6$. Therefore by the triangle inequality

$$6 = |a_N - a_{N+1}| \leq |a_N - \alpha| + |a_{N+1} - \alpha|.$$

Thus we must have that either $|a_N - \alpha| > 1 = \varepsilon$ or $|a_{N+1} - \alpha| > 1$. Therefore we have found $\varepsilon > 0$ such that for all $N \in \mathbb{N}$ there exists $n \geq N$ (in this case either N or $N+1$) with $|a_n - \alpha| > \varepsilon$. Therefore we cannot have $\lim_{n\to\infty} a_n = \alpha$. Since this holds for all $\alpha \in \mathbb{R}$ we must have that (a_n) is a divergent sequence.

(c) Let $\alpha \in \mathbb{R}$. We take $\varepsilon = 1/4$ and let $N \in \mathbb{N}$. If $n \geq 8$ and odd then $a_n < 0$ and if n is even then $a_n = \frac{n+8}{n} \geq 1$. Thus we can choose $n \geq N$ (just take $n = \max\{N, 8\}$) such that $|a_{n+1} - a_n| \geq 1$. Therefore by the triangle inequality

$$1 \leq |a_n - \alpha| + |a_{n+1} - \alpha|$$

and so one of $|a_n - \alpha|$ or $|a_{n+1} - \alpha|$ must be greater than $1/4$. Therefore (a_n) is divergent.

5. Let $\varepsilon > 0$. Since $\lim_{n\to\infty} x_n = x$ we can find $N_1 \in \mathbb{N}$ such that if $n \geq N_1$ then $|x_n - x| \leq \varepsilon$. We choose $N = N_1$. If $n \geq N$ then since $n + k \geq N$ we will have $|x_{n+k} - x| \leq \varepsilon$ and so,

$$|y_n - x| = |x_{n+k} - x| \leq \varepsilon.$$

6. We will show that for all $\varepsilon > 0$ we have that $a - b \leq \varepsilon$ which implies $a \leq b$. So we let $\varepsilon > 0$ and use that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. This means we can choose $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - a| \leq \varepsilon/2$ and $|b_n - b| \leq \varepsilon/2$. Therefore

$$a - b = a - a_N + a_N - b_N + b_N - b \leq |a - a_N| + a_N - b_N + |b_N - b| \leq \varepsilon + a_N - b_N.$$

So $a - b \leq \varepsilon + a_N - b_N$ and since for all $n \in \mathbb{N}$ we have that $a_n \leq b_n$ we can conclude that $a - b \leq \varepsilon$.

Alternatively: Since $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, we know from the Rules of Limits that $\lim_{n\to\infty}(b_n - a_n) = b - a$. We claim that $b - a \geq 0$. For the sake of contradiction, suppose that $b - a < 0$. Take $\varepsilon = (a - b)/2$; so $\varepsilon > 0$. Also, since $\lim_{n\to\infty}(b_n - a_n) = b - a$, there is some $N \in \mathbb{N}$ so that for all $n \geq N$ we have $|b_n - a_n - b + a| \leq \varepsilon$. Since $b_n - a_n \geq 0$ and $a - b > 0$, we have

$$0 < b_n - a_n - b + a = |b_n - a_n - b + a| \leq \varepsilon = \frac{a-b}{2}.$$

Hence $0 \leq b_n - a_n < \frac{a-b}{2} + b - a = \frac{b-a}{2} < 0$, a contradiction since we do not have $0 < 0$. Thus we must have $b - a \geq 0$.

7. Let $\alpha \in \mathbb{R}$ and choose $\varepsilon = a/4$. Let $N \in \mathbb{N}$ (we need to find $n \in \mathbb{N}$ with $|a_n - \alpha| > a/4$.) We can find $K \in \mathbb{N}$ such that for all $k \geq K$ $|(x_{k+1} - x_k) - a| \leq a/8$ and in particular $|x_k - x_{k+1}| \geq 7a/8$. Now let $N \in \mathbb{N}$ and choose $k = \max N, K$. In this case $n \geq N$ and by the triangle inequality we have that

$$7a/8 \leq |x_{n+1} - x_n| \leq |x_{n+1} - \alpha| + |x_n - \alpha|.$$

Thus either $|x_n - \alpha| > a/4 = \varepsilon$ or $|x_{n+1} - \alpha| > a/4 = \varepsilon$. either way we can find $n \in \mathbb{N}$ with $n \geq N$ and $|x_n - \alpha| > \varepsilon$.

8. First of all we need to show that $x \geq 0$. To do this we will show that $x \geq -\varepsilon$ for any $\varepsilon > 0$. So let $\varepsilon > 0$ and choose N such that for all $n \geq N$ we have that $|x_n - x| \leq \varepsilon$ (since $\lim_{n\to\infty} x_n = x$). We have that $x - x_N \geq -\varepsilon$ and $x_N \geq 0$ which means that

$$x = x - x_N + x_N \geq -\varepsilon + 0 = -\varepsilon.$$

Thus $x \geq 0$.

Now suppose that $x = 0$. In this case let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ to be such that for all $n \geq N$, $|x_n| \leq \varepsilon^2$ (which we can do since $\lim_{n\to\infty} x_n = x$). Thus for $n \geq N$ we have that

$$|\sqrt{x_n} - 0| = \sqrt{x_n} \leq \sqrt{\varepsilon^2} = \varepsilon.$$

Thus $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{x}$.

Now suppose that $x > 0$. In this case let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ to be such that for all $n \geq N$, $|x_n - x| \leq \varepsilon\sqrt{x}$ (which we can do since $\sqrt{x} > 0$ and $\lim_{n\to\infty} x_n = x$. Note you will need to do some calculations first before figuring out how to choose N.) Thus for $n \geq N$ we have that

$$\begin{aligned} |\sqrt{x_n} - \sqrt{x}| &= \left| \frac{(\sqrt{x_n} - \sqrt{x})(\sqrt{x_n} + \sqrt{x})}{\sqrt{x_n} + \sqrt{x}} \right| \\ &= \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \leq \frac{|x_n - x|}{\sqrt{x}} \leq \varepsilon. \end{aligned}$$

Thus $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{x}$.

9. (a) We know that for all $n \in \mathbb{N}$, $0 \leq \frac{1}{n^3+5} \leq \frac{1}{n^3}$. Since we know that $\lim_{n\to\infty} 0 = 0$ and $\lim_{n\to\infty} n^{-3} = 0$ (lecture notes) it follows by the sandwich rule that $\lim_{n\to\infty} \frac{1}{n^3+5} = 0$.

   (b) We have that for all $n \in \mathbb{N}$, $\frac{n^2+3}{4n^2+7n} = \frac{1+3n^{-2}}{4+7n^{-1}}$. By the sum and scalar product rules it follows that $\lim_{n\to\infty} 1 + 3n^{-2} = 1$ and $\lim_{n\to\infty} 4 + 7n^{-1} = 4$. It then follows by the quotient rule that $\lim_{n\to\infty} \frac{n^2+3}{4n^2+7n} = \frac{1}{4}$.

   (c) We have that for all $n \in \mathbb{N}$, (using that $1 = (\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})$)

$$0 \leq \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}} \leq \frac{1}{\sqrt{n}}.$$

   It follows by question 8 that $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ (covered in the problems class) and so by the sandwich rule $\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} = 0$.

   (d) We have that for all $n \in \mathbb{N}$

$$\frac{\sin n + 5n}{n^2} = n^{-2}(\sin n) + 5n^{-1}.$$

   We know that since $-1 \leq \sin n \leq 1$, $-n^{-2} \leq n^{-2}\sin n \leq n^{-2}$. Thus by the sandwich rule and the scalar product rule $\lim_{n\to\infty} n^{-2}\sin n = 0$ and so by the sum rule and the scalar product rule $\lim_{n\to\infty} \frac{\sin n+5n}{n^2} = 0$.

10. Since $x < 0$ and $\lim_{n\to\infty} x_n = x$ we can choose N such that for all $n \geq N$ $|x_n - x| \leq -x/2$. Thus $x_n - x \leq -x/2$ and so $x_n \leq x/2 < 0$. Thus there are at most $N-1$ (finite) values of n where $x_n \geq 0$.

11. Since $(b_n)_{n\in\mathbb{N}}$ is bounded there exists $K > 0$ such that for all $n \geq N$ $|b_n| \leq K$. Thus for all $n \in \mathbb{N}$ $0 \leq |a_n b_n| \leq K|b_n|$. So if we let $\varepsilon > 0$ and choose $n \in \mathbb{N}$ where for all $n \geq N$ $|a_n| \leq \varepsilon/K$ then $|a_n b_n| \leq K|b_n| \leq \varepsilon$ and thus $\lim_{n\to\infty} a_n b_n = 0$.

12. Since $|a| > 0$ and $\lim_{n\to\infty} a_n = a$ we can find $N \in \mathbb{N}$ such that for all $n \geq N$ $|a_n - a| \leq |a|/2$ and thus for $n \geq N$, $a_n \neq 0$. So by question 5 from sheet 4 $\lim_{n\to\infty} a_{n+N} = a$ and $a_{n+N} \neq 0$ for all $n \in \mathbb{N}$. Now in order to obtain a contradiction suppose that $(b_n a_n)$ is convergent then there exists $c \in \mathbb{R}$ where $\lim_{n\to\infty} a_n b_n = c$ and thus $\lim_{n\to\infty} a_{n+N} b_{n+N} = c$. So by the quotient rule $\lim_{n\to\infty} b_{n+N} = c/a$ and thus $\lim_{n\to\infty} b_n = c/a$ (let $\varepsilon > 0$, choose N_1 such that for all $n \geq N + N_1$ $|b_{n+N} - c/a| \leq \varepsilon$ and thus if we take $N_2 = N_1 + N$ for all $n \geq N$ we have that $|b_n - c/a| \leq \varepsilon$ and so $\lim_{n\to\infty} b_n = c/a$). Thus (b_n) is convergent which is a contradiction. So $(a_n b_n)$ must be divergent.