Solutions to exercise sheet 5: Sequences divergent to infinity and monotone sequences

1. (a) For all $n \in \mathbb{N}$ we have that

$$\frac{n^2+1}{n+3} = \frac{1+n^{-2}}{n^{-1}+3n^{-2}} \geq \frac{1}{4n^{-1}} = n/4.$$

Thus for any $K \in \mathbb{R}$ by the Archimedean principle we can find $N \in \mathbb{N}$ with $N > 4K$. For $n \geq N$ we have that

$$a_n = \frac{n^2+1}{n+3} \geq n/4 \geq N/4 > K.$$

(b) Let $K \in \mathbb{R}$ by the Archimedean principle we can find $N \in \mathbb{N}$ with $N > K^2$. For $n \geq N$ we have that

$$a_n = \sqrt{n} \geq \sqrt{N} > \sqrt{K^2} \geq K.$$

(c) Note that for any $n \in \mathbb{N}$, $(n+1)^2 - n^2 = 2n+1$. So we let $K \in \mathbb{R}$ and use the Archimedean principle to choose $N \in \mathbb{N}$ such that $N > x$. Thus for any $n \geq N$ we have that

$$a_n = 2n+1 \geq 2N+1 \geq N > K.$$

2. (a) Let $b > 0$. By the binomial theorem we have that for $n \geq 1$,

$$(1+b)^n = \sum_{k=0}^{n} \binom{n}{k} b^k.$$

Thus if $n \geq 2$, since $b > 0$ we have that

$$(1+b)^n \geq \binom{n}{2} b^2 = \frac{n(n-1)b^2}{2}.$$

Actually the inequality is also true when $n = 1$).

(b) By part 1) we have that for any $n \geq 2$,

$$(1+b)^n - n \geq \frac{n(n-1)b^2}{2} - n = n(nb^2/2 - (1+b^2/2)).$$

By the Archimedean principle we choose $N_1 \in \mathbb{N}$ such that $N_1 > \frac{2(2+b^2/2)}{b^2}$. Thus for $n \geq N_1$ we have that

$$(1+b)^n - n \geq n.$$

Thus if we let $K \in \mathbb{R}$ we can use the Archimedean principle to choose $N \in \mathbb{N}$ such that $N > \max\{K, N_1\}$. If $n \geq N$ then

$$(1+b)^n - n \geq n \geq N > K.$$

So $\lim_{n\to\infty}(1+b)^n - n = \infty$.

(c) Let $\varepsilon > 0$. By the previous part we can choose $N \in \mathbb{N}$ such that for all $n \geq N$

$$(1+\varepsilon)^n - n \geq 0$$

and so $(1+\varepsilon)^n \geq n$. Thus for $n \geq N$,

$$1 \leq n \leq (1+\varepsilon)^n$$

and so

$$1 \leq n^{1/n} \leq 1+\varepsilon.$$

Thus $|n^{1/n} - 1| \leq \varepsilon$.

3. Let $c \in (0,\infty)$ and $K \in \mathbb{R}$. Since $\lim a_n = \infty$ we can find $N \in \mathbb{N}$ such that for all $n \geq N$, $a_n > \frac{K}{c}$. Thus for $n \geq N$ we will have that

$$ca_n > c\frac{K}{c} = K.$$

Similarly for $n \geq N$ we have that

$$-ca_n < -c\frac{K}{c} = -c.$$

Thus $\lim_{n\to\infty} ca_n = \infty$ and $\lim_{n\to\infty} -ca_n = -\infty$.

4. (a) Let $K \in \mathbb{R}$. Since $\lim_{n\to\infty} b_n = b > 0$ we can find $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $|b_n - b| \leq b/2$ and so $b_n \geq b/2$. Since $\lim_{n\to\infty} a_n = \infty$, we can find $N_2 \in \mathbb{N}$ such that for all $n \geq N$, $a_n > \max\{2K/b, 0\}$. Now let $N = \max\{N_1, N_2\}$. For $n \geq N$ we have that since $a_n > 0$, $a_n > 2K/b$ and $b_n \geq b/2$

$$a_n b_n \geq a_n(b/2) > K.$$

Thus $\lim_{n\to\infty} a_n b_n = \infty$.

(b) Let $K \in \mathbb{R}$. Since $\lim_{n\to\infty} b_n = b < 0$ we can find $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $|b_n - b| \leq -b/2$ and so $b_n \leq b/2$. Since $\lim_{n\to\infty} a_n = \infty$, we can find $N_2 \in \mathbb{N}$ such that for all $n \geq N$, $a_n > \max\{-2K/b, 0\}$. Now let $N = \max\{N_1, N_2\}$. For $n \geq N$ we have that since $a_n > 0$, $a_n > -2K/b$ and $b_n \leq b/2$,

$$a_n b_n \leq a_n(b/2) < K.$$

Thus $\lim_{n\to\infty} a_n b_n = -\infty$.

(c) i. Take $a_n = n^2$ and $b_n = n^{-1}$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} n^2 = \infty$, $\lim_{n\to\infty} n^{-1} = 0$ and $a_n b_n = n$ for all $n \in \mathbb{N}$ so $\lim_{n\to\infty} a_n b_n = \infty$.

ii. Take $a_n = n$, $b_n = n^{-2}$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} n^2 = \infty$, $\lim_{n\to\infty} n^{-1} = 0$ and $a_n b_n = n^{-1}$ for all $n \in \mathbb{N}$ so $\lim_{n\to\infty} a_n b_n = 0$.

iii. Take $a_n = n^2$ and $b_n = -n^{-1}$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} n^2 = \infty$, $\lim_{n\to\infty} -n^{-1} = 0$ and $a_n b_n = -n$ for all $n \in \mathbb{N}$ so $\lim_{n\to\infty} a_n b_n = -\infty$.

5. Let $K \in \mathbb{R}$, since $\lim_{n\to\infty} a_n = \infty$ there exists $N_1 \in \mathbb{N}$ such that if $n \in \mathbb{N}$ and $n \geq N_1$ then $a_n > K$. Now let $N_2 = \max\{N_1 + k, N + k\}$. If $n \geq N_2$ then since $n - k \geq N$ we have that $b_n \geq a_{n-k}$. Since we also have that $n - k \geq N_1$ we also know that $a_{n-k} > K$. Thus

$$b_n \geq a_{n-k} > K$$

and we can conclude that $\lim_{n\to\infty} b_n = \infty$.

6. Let $(b_n)_{n\in\mathbb{N}}$ be the sequence where $b_n = -a_n$ for all $n \in \mathbb{N}$. We then have that since $a_{n+1} \leq a_n$, $b_n = -a_n \leq -a_{n+1} = b_{n+1}$ for all $n \in \mathbb{N}$. Therefore (b_n) is monotone increasing. Moreover there exists $K \in \mathbb{R}$ such that $a_n \geq K$ for any $n \in \mathbb{N}$ and so $b_n = -a_n \leq -K$ for all $n \in \mathbb{N}$. Thus (b_n) is monotone increasing and bounded above and so convergent. By the scalar product rule since $a_n = -b_n$ for all $n \in \mathbb{N}$ we must have that (a_n) is convergent.

7. Since $(a_n)_{n\in\mathbb{N}}$ is bounded there exists $K \in \mathbb{R}$ such that $a_n \leq K$ for all $n \in \mathbb{N}$. Therefore for all $k \in \mathbb{N}$ $a_k \leq K$ and so

$$b_n = \inf\{a_k : k \geq n\} \leq a_k \leq K.$$

Moreover for all $n \in \mathbb{N}$,

$$\{a_k : k \geq n+1\} \subseteq \{a_k : k \geq n\}$$

and so by question 2 on exercise sheet 3 $b_n \leq b_{n+1}$. Thus (b_n) is a monotone increasing sequence which is bounded above and so (b_n) is convergent.

8. First of all note that since $a_1 \leq a_n$ for all $n \in \mathbb{N}$ (a_n) is bounded below and so must be unbounded above. Let $x \in \mathbb{R}$. Since (a_n) is unbounded above we can choose $N \in \mathbb{N}$ such that $a_N > x$. Since a_n is monotone increasing for all $n \geq N$ we have that $a_n \geq a_N > x$. Thus $\lim_{n\to\infty} a_n = \infty$.

9. (a) We will show that this is true by induction. It is clearly true when $n = 1$ since $0 \leq 2 \leq 1 + \sqrt{2}$. Now suppose for some $k \in \mathbb{N}$ we have that $0 \leq a_k \leq 1 + \sqrt{2}$. We then have that $a_{k+1} = \frac{5a_k+2}{2a_k+1} \geq 0$. Moreover we have that

$$5a_k + 2 - (\sqrt{2}+1)(2a_k+1) = a_k(3 - 2(\sqrt{2})) + 1 - \sqrt{2}.$$

Since $0 \leq a_k \leq 1+\sqrt{2}$ and $3-2\sqrt{2} > 0$ we can conclude that

$$\begin{aligned} 5a_k + 2 - (\sqrt{2}+1)(2a_k+1) &\leq (1+\sqrt{2})(3-2(\sqrt{2})+1-\sqrt{2} \\ &= -1+\sqrt{2}+1-\sqrt{2} = 0. \end{aligned}$$

Thus

$$5a_k + 2 - (\sqrt{2}+1)(2a_k+1) \leq 0$$

and so

$$a_{k+1} = \frac{5a_k+2}{2a_k+1} \leq 1+\sqrt{2}.$$

Therefore we can conclude by induction that $0 \leq a_n \leq 1+\sqrt{2}$ for all $n \in \mathbb{N}$.

(b) We will show that (a_n) is monotone increasing. We have that for all $n \in \mathbb{N}$

$$\begin{aligned} a_{n+1} - a_n &= \frac{5a_n+2}{2a_n+1} - a_n \\ &= \frac{-2a_n^2+4a_n+2}{2a_n+1}. \end{aligned}$$

We have that since $0 \leq a_n \leq 1+\sqrt{2}$,

$$= -2a_n^2+4a_n+2 = -2(a_n^2-2a_n-1) = -2(a_n-(1+\sqrt{2}))(a_n-(1-\sqrt{2})) \geq 0$$

and so $a_{n+1} - a_n \geq 0$. Therefore (a_n) is a monotone increasing sequence which is bounded above and is thus convergent.

(c) We know (a_n) is convergent so we can find $\alpha \in \mathbb{R}$ with $\lim_{n\to\infty} a_n = \alpha$. Since for all $n \in \mathbb{N}$ $a_n \geq 1$ we know that $\alpha > 1$ and so by the arithmetic properties of limits

$$\alpha = \lim_{n\to\infty} \frac{5a_n+2}{2a_n+1} = \frac{5\alpha+2}{2\alpha+1}.$$

Therefore α satisfies that $2\alpha^2 - 4\alpha - 2 = 0$ and so $\alpha = 1 \pm \sqrt{2}$. Since we know that $\alpha \geq 1$ we must have that $\alpha = 1+\sqrt{2}$.

10. Fix $n \in \mathbb{N}$. We know that since $(a_{n+1}, b_{n+1}) \subseteq (a_n, b_n)$ that $a_{n+1} \geq a_n$ and $b_{n+1} \leq b_n$. Therefore (a_n) is monotone increasing and (b_n) is monotone decreasing. We also know that for $n \in \mathbb{N}$, $a_n \leq b_n \leq b_1$ and $b_n \geq a_n \geq a_1$. Thus (a_n) is bounded above and (b_n) is bounded below. Therefore by the monotone convergence theorem (a_n) and (b_n) are convergent.