Solutions to exercise sheet 6: Subsequences and Cauchy sequences

1. (a) This statement is false. Consider the sequence $a_n = (-1)^n$ for all $n \in \mathbb{N}$. This has a convergent subsequence $(a_2 n)_{n\in\mathbb{N}}$ but the sequence itself is not convergent.

   (b) This statement is true since if (a_n) is convergent then all subsequence of (a_n) must be convergent. So if (a_n) has a divergent subsequence it must be divergent.

   (c) This statement is true. Suppose (a_n) has a bounded subsequence $(a_{n_k})_{k\in\mathbb{N}}$ then by the Bolzano Weierstrass Theorem $(a_{n_k})_{k\in\mathbb{N}}$ will have a convergent subsequence which will also be a subsequence of $(a_n)_{n\in\mathbb{N}}$.

   (d) This statement is false. Question 3 (c) gives a counter example.

2. (a) Since $(|a_n|)_{n\in\mathbb{N}}$ is not divergent to infinity this means that there exists $x \in \mathbb{R}$ such that for all $N \in \mathbb{N}$ there exists $n \in \mathbb{N}$ with $n \geq N$ and $|a_n| \leq x$. Thus we can build a bounded subsequence of $(|a_n|)$ as follows. First of all take $N = 1$ and choose $n_1 \geq 1$ such that $|a_{n_1}| \leq x$. For $k \in \mathbb{N}$ take $N = n_k + 1$ and choose $n_{k+1} \geq N$ such that $|a_{n_k}| \leq x$. Therefore we have a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ such that $|a_{n_k}| \leq x$ for all $k \in \mathbb{N}$ and so it is bounded. Thus by the Bolzano-Weierstrass Theorem a_{n_k} has a subsequence which is convergent and therefore (a_n) has a subsequence which is convergent.

   (b) First of all suppose that $\lim_{n\to\infty} |a_n| = \infty$. Then by Proposition 5.6 form the lecture notes we know that any subsequence (a_{n_k}) of (a_n) must satisfy that $\lim_{k\to\infty} |a_{n_k}| = \infty$ and is therefore not convergent. On the other hand if $(a_n)_{n\in\mathbb{N}}$ does not have a convergent subsequence then by the first part we must have that $\lim_{n\to\infty} |a_n| = \infty$.

3. (a) We have that for all $n \in \mathbb{N}$

$$\left|\frac{n(-1)^n + 7}{n+5}\right| \leq \frac{n+7}{n+5} = 1 + \frac{2}{n+5} \leq \frac{4}{3}.$$

   So (a_n) is bounded and so by the Bolzano-Weierstrass Theorem has a convergent subsequence.

   (b) We have that for all $n \in \mathbb{N}$

$$\frac{n^2+5}{n+4} \geq \frac{n^2}{n+4} \geq \frac{n^2}{5n} = n/5$$

   and so $\lim_{n\to\infty} a_n = \infty$ and thus (a_n) thus not have any convergent subsequences.

(c) Note that $a_{2n+1} = -(2n+1) + 2n + 1 = 0$ for all $n \in \mathbb{N}$. So $(a_{2n+1})_{n\in\mathbb{N}}$ is convergent and thus (a_n) has a convergent subsequence.

(d) We have that $|a_n| \geq n^2 - n$ for all $n \in \mathbb{N}$ and so $\lim_{n\to\infty} |a_n| = \infty$ and thus (a_n) does not have a convergent subsequence.

4. (a) Let $\varepsilon > 0$ choose $N \in \mathbb{N}$ such that $3^{-N} \leq \varepsilon/2$ (which we can do since $\lim_{n\to\infty} 3^{-n} = 0$). For all $n, m \geq N$ we have that

$$\left|\frac{3^n+1}{3^n} - \frac{3^m+1}{3^m}\right| = \left|\frac{3^m - 3^n}{3^n 3^m}\right| \leq \frac{1}{3^m} + \frac{1}{3^n} \leq 2 \cdot 3^{-N} \leq \varepsilon.$$

(b) Let $\varepsilon > 0$ choose $N \in \mathbb{N}$ such that $N \geq \frac{2}{\varepsilon}$ which we can do by the Archimedean principle. For $n, m \geq N$

$$\left|\frac{5+n}{7n} - \frac{5+m}{7m}\right| = \left|\frac{35(m-n)}{49nm}\right| \leq \frac{35}{49n} + \frac{35}{49m} \leq \frac{2}{N} \leq \varepsilon.$$

(c) We will show by induction that $|a_{n+1} - a_n| = 2^{-n}$ for all $n \in \mathbb{N}$. This is certainly true for $n = 1$. If we suppose it is true for $n = k$ then

$$|a_{k+2} - a_{k+1}| = |1 - a_{k+1}/2 - (1 - a_k/2)| = |a_{k+1} - a_k|/2 = 2^{-k+1}.$$

So by induction we have that $|a_{n+1} - a_n| = 2^{-n}$ for all $n \in \mathbb{N}$. Now let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that $2^{-N} \leq \varepsilon/2$ (which we can do since $\lim_{n\to\infty} 2^{-n} = 0$). For $n > m \geq N$ we have that

$$|a_n - a_m| \leq \sum_{k=m}^{n-1} |a_{k+1} - a_k| = \sum_{k=m}^{n-1} 2^{-k} \leq 2^{-m+1} \leq 2^{-N+1} \leq \varepsilon.$$

5. Let $\varepsilon > 0$. Choose N such that for all $n, m \geq N$ $|a_n - a_m| \leq \varepsilon/2$ and $|b_n - b_m| \leq \varepsilon/2$. Thus for $n, m \geq N$

$$|a_n + b_n - (a_m + b_m)| \leq |a_n - a_m| + |b_n - b_m| \leq \varepsilon.$$

6. (a) This statement is true since by the Bolzano-Weierstrass Theorem any bounded sequence contains a convergent subsequence which will be Cauchy since all convergent sequences are Cauchy.

(b) This is not true. Since $\lim_{n\to\infty} a_n = \infty$ it follows that for all subsequences $(n_k)_{k\in\mathbb{N}}$, $\lim_{k\to\infty} a_{n_k} = \infty$ and so all subsequences are divergent and so are not Cauchy.

(c) This is not true. The sequence $(a_n)_{n\in\mathbb{N}}$ where $a_n = n$ for all $n \in \mathbb{N}$ is monotone increasing but $|a_{n+1} - a_n| = 1$ for all $n \in \mathbb{N}$ and so the sequence cannot be Cauchy.

7. Consider the sequence with terms $b_n = \sum_{k=1}^{n} |a_{k+1} - a_k|$. $0 \le b_n \le M$ for all $n \in \mathbb{N}$ so the sequence is bounded and $b_{n+1} - b_n \ge 0$ so the sequence is monotone increasing. Therefore (b_n) is convergent and thus Cauchy. Now let $\varepsilon > 0$, choose N such that for all $n, m \ge N + 1$, $|b_n - b_m| \le \varepsilon$. Thus for $n, m \ge N$ with $m > n$,

$$|a_n - a_m| \le \sum_{k=n}^{m-1} |a_{k+1} - a_k| \le |b_{m-1} - b_{n-1}| \le \varepsilon.$$

8. (a) We know that $a_1 \ge 2$. We now suppose that for some $k \in \mathbb{N}$, $a_k \ge 2$. We then have that $a_{k+1} \ge 2 + \frac{1}{a_n} \ge 2$. Thus by induction $a_n \ge 2$ for all $n \in \mathbb{N}$.

   (b) We have for any $n \in \mathbb{N}$ with $n \ge 2$,

$$|a_{n+1} - a_n| = \left| \frac{1}{a_n} - \frac{1}{a_{n-1}} \right| = \left| \frac{a_{n-1} - a_n}{a_n a_{n-1}} \right| \le \frac{1}{4} |a_n - a_{n-1}|.$$

   (c) We know that $|a_2 - a_1| = \frac{1}{2}$. So by induction and using the formula above we can show that $|a_{k+1} - a_k| \le \frac{4^{-(k-1)}}{2}$. Thus for all $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} |a_{k+1} - a_k| \le \sum_{k=1}^{n} \frac{1}{2 \cdot 4^{k-1}} \le 1.$$

   So by question 7 we can conclude that (a_n) is Cauchy and thus convergent. Now let $\alpha = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$. We know that since $a_n \ge 2$ for all $n \in \mathbb{N}$ we have that $\alpha \ge 2$. Thus by the quotient rule and the sum rule for limits of convergent sequences $\alpha = 2 + \alpha^{-1}$ and so $\alpha^2 - 2\alpha - 1 = 0$. Thus $\alpha = 1 \pm \sqrt{2}$ and since $\alpha \ge 2$ we must have that $\alpha = 1 + \sqrt{2}$.

9. (a) We have that $a_1 \ge 1$ and if $a_n \ge 1$ then $a_{n+1} = 1 + \frac{1}{a_n} \ge 1$ and so $a_n \ge 1$ for all $n \in \mathbb{N}$ by induction. Thus if (a_n) has a limit then it must be greater than 1. Suppose $(a_n)_{n\in\mathbb{N}}$ is convergent and $\lim_{n\to\infty} a_n = \alpha \ge 1$. By the quotient rule and sum rule for limits of sequences $\alpha^2 = \alpha + 1$ and so $\alpha = \frac{1\pm\sqrt{5}}{2}$ but since $\alpha \ge 1$ we must have that $\alpha = \frac{1+\sqrt{5}}{2}$.

   (b) We have that since $1 = \alpha^2 - \alpha = \alpha(\alpha - 1)$

$$\begin{aligned} a_{n+1} - \alpha &= 1 + \frac{1}{a_n} - \alpha = \frac{a_n + 1 - \alpha a_n}{a_n} \\ &= \frac{(1-\alpha)a_n - 1}{a_n} = \frac{(1-\alpha)a_n - (\alpha - 1)\alpha}{a_n}. \end{aligned}$$

   Thus since $a_n \ge 1$, we have that $|a_{n+1} - a_n| \le (\alpha - 1)|a_n - \alpha|$.

(c) We will use the previous part to show that by induction $|a_n - \alpha| \leq (\alpha - 1)^{n-1}$. It is certainly true for $n = 1$ and if for $k \in \mathbb{N}$ we have $|a_k - \alpha| \leq (\alpha - 1)^(k - 1)$ then by part (b), $|a_{k+1} - \alpha| \leq (\alpha - 1)^k$. Thus by induction for all $n \in \mathbb{N}$, $|a_n - \alpha| \leq (\alpha - 1)^{n-1}$. Now let $\varepsilon > 0$. Since $0 < \alpha - 1 < 1$ we can choose N such that for all $n \geq N$ $(\alpha - 1)^{n-1} \leq \varepsilon$. Therefore for $n \geq N$, $|a_n - \alpha| \leq (\alpha - 1)^{n-1} \leq \varepsilon$.