Solutions to exercise sheet 7: Series

1. We have that $S_n = \sum_{k=1}^{n} b_n = \sum_{k=1}^{n} \frac{a_k}{2^k}$. We show that (S_n) is a Cauchy sequence. Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that $2^{-N} \leq \varepsilon$ which we can do since $\lim_{n\to\infty} 2^{-n} = 0$. If $n, m \in \mathbb{N}$ with $m > n \geq N$ then we have that

$$\begin{aligned} |S_m - S_n| &= \left|\sum_{k=n+1}^{m} b_k\right| \leq \sum_{k=n+1}^{m} |b_k| \\ &= \sum_{k=n+1}^{m} \frac{|a_k|}{2^k} = \sum_{k=n+1}^{m} 2^{-k} \leq 2^{-n} \\ &\leq 2^{-N} \leq \varepsilon. \end{aligned}$$

Thus (S_n) is a Cauchy sequence and so convergent. This means the series $\sum_{n=1}^{\infty} b_n$ is convergent.

2. This is essentially the same question as question 7 from the previous sheet (exercise sheet 6) in a slightly different form. Since the series $\sum_{n=1}^{\infty} |a_{n+1} - a_n|$ is convergent we know that the sequence of partial sums $\sum_{n=1}^{k} a_n$ is bounded. Therefore there exists $M > 0$ such that for all $k \in \mathbb{N}$, $\sum_{n=1}^{k} |a_{n+1} - a_n| \leq M$ therefore by question 7 from exercise sheet 6 we know that the sequence $(a_n)_{n\in\mathbb{N}}$ is convergent.

3. We have that
$$\frac{2}{n(n+2)} = \frac{1}{n} - \frac{1}{(n+2)}.$$
Thus if we compute the partial sums, S_k we get that for $k \geq 2$
$$S_k = \sum_{n=1}^{k} \frac{2}{n(n+2)} = \sum_{n=1}^{k} \frac{1}{n} - \sum_{n=1}^{k} \frac{1}{n+2} = 1 + \frac{1}{2} - \frac{1}{k+1} - \frac{1}{k+2}.$$
Hence using the sum rule for sequences $\lim_{k\to\infty} S_k = \frac{3}{2}$ and so the series is convergent and $\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \frac{3}{2}$.

4. Since $(b_n)_{n\in\mathbb{N}}$ is a convergent sequence it is bounded and so since the terms are all non-negative there exists $K > 0$ such that for all $n \in \mathbb{N}$, $0 \leq b_n \leq K$. Therefore since for all $n \in \mathbb{N}$, $a_n \geq 0$ we get that $0 \leq a_n b_n \leq K a_n$. By a result in lectures (Proposition 6.4) since the series $\sum_{n=1}^{\infty} a_n$ is convergent the series $\sum_{n=1}^{\infty} K a_n$ is convergent. Therefore by the comparison test the series $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

5. We know that for all $k \in \mathbb{N}$, $0 < \frac{a_k}{n^k} \leq \frac{n-1}{n^k}$. Therefore we get that for all $j \in \mathbb{N}$, the partial sums of this series satisfy,
$$S_k \leq \sum_{k=1}^{j} \frac{a_k}{n^k} \leq \sum_{k=1}^{j} \frac{n-1}{n^k} \leq 1 - n^{-k-1} \leq 1.$$

Thus the sequence of partial sums is a bounded sequence and since all the terms in the series are positive it is increasing and therefore it is convergent. This can also be solved using the comparison test and the result on when geometric series converge. Note that if $n = 10$ this series would give by the decimal expansion of a number between 0 and 1.

6. (a) $\lim_{n\to\infty} \frac{n}{n+6} = 1 \neq 0$ and so this series is divergent.

   (b) We have that for all $n \in \mathbb{N}$, $0 \leq \frac{1}{n^2+3n+4} \leq \frac{1}{n^2}$ and since we know that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent we can conclude by the comparison test that this series is convergent.

   (c) We can write
$$\frac{1}{n^2 - 6n + 3} = \frac{1}{n^2} \frac{1}{1 - 6n^{-1} + 3n^{-2}}.$$
   Since by the arithmetic properties of limits of sequences $\lim_{n\to\infty} \frac{1}{1-6n^{-1}+3n^{-2}} = 1$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$,
$$\frac{3}{2} \geq \frac{1}{1 - 6n^{-1} + 3n^{-2}} \geq 0$$
   and therefore
$$0 \leq \frac{1}{n^2 - 6n + 3} \leq \frac{3}{2n^2}.$$
   Since a constant multiple of a convergent sequence is convergent we know that the series $\sum_{n=1}^{\infty} \frac{3}{2n^2}$ is convergent and therefore by the comparison test we know that the series $\sum_{n=1}^{\infty} \frac{1}{n^2-6n+3}$ is convergent.

   (d) We can write
$$\frac{n}{n^2 - 6n + 3} = \frac{1}{n} \frac{1}{1 - 6n^{-1} + 3n^{-2}}.$$
   By the arithmetic properties of limits of sequences we have that $\lim_{n\to\infty} \frac{1}{1-6n^{-1}+3n^{-2}} = 1$. Thus there exists $N \in \mathbb{N}$ such that for all $n \geq N$,
$$\frac{n}{n^2 - 6n + 3} \geq \frac{1}{2n} \geq 0$$
   Now the series $\sum_{n=1}^{\infty} \frac{1}{2n}$ is divergent since the harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. Therefore the series $\sum_{n=1}^{\infty} \frac{n}{n^2-6n+3}$ is divergent.

7. (a) Fix $r \in (\lambda, 1)$. Since $\lim_{n\to\infty} a_n^{1/n} = \lambda$ we can find r such that for all $n \geq N$, $a_n^{1/n} < r$. Therefore for $n \geq N$, $0 < a_n < r^n$ and since $r < 1$ by the comparison test and the fact that the geometric series $\sum_{n=1}^{\infty} r^n$ is convergent, the series $\sum_{n=1}^{\infty} a_n$ is convergent.

(b) Fix $r \in (1, \lambda)$. Since $\lim_{n\to\infty} a_n^{1/n} = \lambda$ we can find $N \in \mathbb{N}$ such that for all $n \geq N$, $a_n^{1/n} > r$. Therefore for $n \geq N$, $0 \leq r^n \leq a_n$ and by the comparison test and the fact that the geometric series $\sum_{n=1}^{\infty} r^n$ is divergent, the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(c) For an example when the series is divergent we simply need to take $a_n = 1$ for all $n \in \mathbb{N}$. In this case $a_n^{1/n} = 1$ for all $n \in \mathbb{N}$ and so $\lim_{n\to\infty} a_n^{1/n} = 1$. However the series $\sum_{n=1}^{\infty} 1$ is divergent.
For an example when the series is divergent we can take $a_n = n^{-2}$ for all $n \in \mathbb{N}$. In this case $a_n^{1/n} = n^{-2/n}$. So we need to show that $\lim_{n\to\infty} n^{-2/n} = 1$. We can use that $\lim_{n\to\infty} n^{1/n} = 1$ (see sheet 5 question 2 (c)) which means by the product rule $\lim_{n\to\infty} n^{2/n} = 1$ and so by the quotient rule $\lim_{n\to\infty} n^{-2/n} = 1$.

8. (a) Let $a_n = \frac{6^n}{n!}$. We have that $\frac{a_{n+1}}{a_n} = \frac{6}{n+1}$ and so $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0$. Thus by the ratio test we have that $\sum_{n=1}^{\infty} \frac{6^n}{n!}$ is convergent.

(b) Let $a_n = \frac{n^6}{2^n}$. We have that $\frac{a_{n+1}}{a_n} = \frac{2(n+1)^6}{n^6}$ and so $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 2$. Thus by the ratio test we have that $\sum_{n=1}^{\infty} \frac{2^n}{n^6}$ is divergent. You could also show that (a_n) is a divergent sequence so the series $\sum_{n=1}^{\infty} a_n$ must be divergent.

(c) Let $a_n = \frac{n}{2^n}$. We have that $\frac{a_{n+1}}{a_n} = \frac{n+1}{2n}$ and so $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{2}$. Thus by the ratio test we have that $\sum_{n=1}^{\infty} \frac{n}{2^n}$ is convergent.

(d) We let S_n be the partial sums. We can calculate

$$S_n = \sum_{k=1}^{n} (\sqrt{k+1} - \sqrt{k}) = \sqrt{n+1} - 1.$$

So (S_n) is a divergent sequence and so the series $\sum_{k=1}^{\infty} (\sqrt{k+1} - \sqrt{k})$ is divergent.

(e) We let $a_n = \frac{n!+n}{(n+2)!} = \frac{1}{(n+1)(n+2)} + \frac{n}{(n+2)!}$. We know that both the series $\sum_{k=1}^{\infty} \frac{1}{(k+1)(k+2)}$ and $\sum_{k=1}^{\infty} \frac{1}{k!}$ are convergent. The first is covergent by the comparison test since $0 \leq \frac{1}{(k+1)(k+2)} \leq \frac{1}{k^2}$ and the fact that the second series is convergent follows by the ratio test. Since the sum of two convergent series is convergent the series $\sum_{n=1}^{\infty} \frac{n!+n}{(n+2)!}$ is convergent.

9. (a) Let $x \in (0, \infty)$ and $a_n = \frac{x^{(n-1)}}{(n-1)!}$. We have that $\frac{a_{n+1}}{a_n} = \frac{x}{n}$ and since $\lim_{n\to\infty} \frac{x}{n} = 0$ we can see that the series $\sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$ is convergent for any $x \in (0, \infty)$.

(b) This series is simply the geometric series, $\sum_{n=1}^{\infty} ar^{n-1}$ where $a = x/2$ and $r = x/2$. So it is convergent whenever $x \in (0, 2)$ but not for any $x > 2$.