Solutions to exercise sheet 8: Limits and continuity of functions

1. (a) In the definition that $\lim_{x\to\infty} f(x) = a > 0$ we take $\varepsilon = a/2 > 0$ and thus we know that there exists $y \in (0,\infty)$ such that if $x \geq y$ then $|f(x) - a| \leq a/2$ and therefore $a/2 \leq f(x) \leq 3a/2$. Since $a > 0$ we know that $f(x) > 0$ for all $x \geq y$.

   (b) We will show that for all $\varepsilon > 0$ we have that $a \leq b + \varepsilon$ and so $a \leq b$. Let $\varepsilon > 0$ and choose $y \in (0,\infty)$ such that if $x \geq y$ then $|f(x) - a| \leq \varepsilon/2$ and $|g(x) - b| \leq \varepsilon/2$ (we can do this since $\lim_{x\to\infty} f(x) = a$ and $\lim_{x\to\infty} g(x) = b$.) Thus, since $f(x) \leq g(x)$,

$$\begin{aligned} a - b &= a - f(x) + f(x) - g(x) + g(x) - b \\ &\leq |a - f(x)| + f(x) - g(x) + |g(x) - b| \leq \varepsilon/2 + 0 + \varepsilon/2 = \varepsilon. \end{aligned}$$

   So $a \leq b+\varepsilon$ and since this holds for all $\varepsilon > 0$ we have that $a \leq b$.

2. We will first show that $\lim_{x\to\infty} \frac{x-3}{x} = 1$. Let $\varepsilon > 0$ and choose $r = 3/\varepsilon$. If $x \geq r$ then

$$\left|\frac{x-3}{x} - 1\right| = \frac{3}{x} \leq \frac{3}{r} = \varepsilon.$$

   Now we show that $\lim_{x\to 0} \frac{x-3}{x} = -\infty$. We need to show that for all $y \in \mathbb{R}$ then there exists $\delta > 0$ such that for all $x \in (0,\delta)$ we have that $\frac{x-3}{x} \leq y$. If $y \geq 1$ then we can take $\delta = 1$ since for all $x \in (0,1)$ we have that $f(x) = (x-3)/x \leq 1$. If $y < 1$ then we choose $\delta = \frac{3}{1-y}$. In this case if $x \geq r$,

$$f(x) = 1 - \frac{3}{x} \leq 1 - \frac{3}{\frac{3}{1-y}} = 1 - (1-y) = y.$$

3. For all parts of this question we will use the Heine definition of the limit of a function.

   (a) Let (x_n) be a sequence such that $x_n \in A$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = a$. Therefore since $\lim_{x\to a} f(a) = b$ and $\lim_{x\to a} g(a) = c$ we know that $\lim_{n\to\infty} f(x_n) = b$ and $\lim_{n\to\infty} g(x_n) = c$. Thus by the sum rule for limits of sequences $\lim_{n\to\infty}(f(x_n) + g(x_n)) = b + c$ and so $\lim_{x\to a}(f + g)(x) = a + b$.

   (b) Let (x_n) be a sequence such that $x_n \in A$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = a$. Therefore since $\lim_{x\to a} f(a) = b$ and $\lim_{x\to a} g(a) = c$ we know that $\lim_{n\to\infty} f(x_n) = b$ and $\lim_{n\to\infty} g(x_n) = c$. Thus by the produce rule for sequences $\lim_{n\to\infty} f(x_n)g(x_n) = bc$ and so $\lim_{x\to a}(f \cdot g)(x) = bc$.

(c) Let (x_n) be a sequence such that $x_n \in A$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = a$. Therefore since $\lim_{x\to a} f(a) = b$ and $\lim_{x\to a} g(a) = c$ we know that $\lim_{n\to\infty} f(x_n) = b$ and $\lim_{n\to\infty} g(x_n) = c$. Thus since $c \neq 0$ and $g(x_n) \neq 0$ for all $n \in \mathbb{N}$ by the quotient rule for sequences we know that

$$\lim_{n\to\infty} \frac{f(x_n)}{g(x_n)} = \frac{b}{c}$$

and so $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{b}{c}$.

4. (This is the direct proof from the definition you could also use the Heine definition). Let $\varepsilon > 0$. Thus we can choose δ such that if $x \in A$ and $0 < |x - a| < \delta$ then $|f(x) - b| \leq \varepsilon$. Therefore by the reverse triangle inequality (see exercise sheet 2 question 3(a)) if $\in A$ and $0 < |x - a| < \delta$ then

$$||f(x)| - |b|| \leq |f(x) - b| \leq \varepsilon.$$

5. (a) Let $\alpha = \sup\{f(x) : x \in \mathbb{R}\}$. Let $\varepsilon > 0$. By the definition of α there exists $y \in \{f(x) : x \in \mathbb{R}\}$ such that $y > \alpha - \varepsilon$. Therefore there exists $r \in \mathbb{R}$ such that $f(r) = y > \alpha - \varepsilon$. If $x \geq r$ then $\alpha \geq f(x) \geq f(r) \geq \alpha - \varepsilon$ and so $|f(x) - \alpha| \leq \varepsilon$. Therefore $\lim_{x\to\infty} f(x) = \alpha$.

   (b) We now know that f is unbounded above. We show that $\lim_{x\to\infty} f(x) = \infty$. Let $k \in \mathbb{R}$ since f is unbounded we can choose $r \in \mathbb{R}$ such that $f(r) \geq k$. Now if $x \geq r$ then since f is increasing we have that $f(x) \geq f(r) \geq k$. Thus $\lim_{x\to\infty} f(x) = \infty$.

6. (a) First of all we know that $|f(0)| \leq 0$ and so $f(0) = 0$. Now let (x_n) be a real valued sequence with $\lim_{n\to\infty} x_n = 0$ and $x_n \neq 0$ for all $n \in \mathbb{N}$. We then have that $|f(x_n)| \leq |x_n|$ which means that $-|x_n| \leq f(x_n) \leq |x_n|$ and since $\lim_{n\to\infty} |x_n| = 0$ we have by the sandwich rule that $\lim_{n\to\infty} f(x_n) = 0$. Thus by the Heine definition $\lim_{x\to 0} f(x) = 0$ and so f continuous at 0.

   (b) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if} \quad x \in \mathbb{Q} \\ 0 & \text{if} \quad x \notin \mathbb{Q} \end{cases}$$

   By part (a) since $|f(x)| \leq |x|$ for all $x \in \mathbb{R}$ we know that f is continuous at 0. On the other hand if we let $x \in \mathbb{Q}\backslash\{0\}$ then we can find a sequence of irrational numbers (x_n) where $\lim_{n\to\infty} x_n = x$. However $\lim_{n\to\infty} f(x_n) = 0 \neq x$ and so f is discontinuous at x.

Now let $x \notin \mathbb{Q}$. Again we can find a sequence of rationals (Lenna 7.16 from the lecture notes) with $\lim_{n\to\infty} x_n = x$. We then have that $f(x_n) = x_n$ for all $n \in \mathbb{N}$ and so $\lim_{n\to\infty} f(x_n) = x \neq 0$. So f is discontinuous at x.

7. Since f, g are both continuous at 0, by taking $\varepsilon = \frac{1}{4}$ in the ε, δ definition of continuity we know there exists $\delta > 0$ such that if $x \leq \delta$ then $|f(0) - f(x)| \leq \frac{1}{4}$ and $|g(0) - g(x)| \leq \frac{1}{4}$. Thus if $x \leq \delta$ then

$$\begin{aligned} f(x) - g(x) &= f(x) - f(0) + f(0) - g(0) + g(0) - g(x) \\ &\geq -|f(x) - f(0)| + f(0) - g(0) - |g(x) - g(0)| \\ &\geq -\frac{1}{4} + 1 - \frac{1}{4} = \frac{1}{2}. \end{aligned}$$

8. For all these questions let (x_n) be a sequence of numbers in a with $\lim_{n\to\infty} x_n = a$. Since f and g are continuous we know that $\lim_{n\to\infty} f(x_n) = f(a)$ and $\lim_{n\to\infty} g(x_n) = g(a)$.

   (a) By the sum rule for sequences we have that $\lim_{n\to\infty}(f+g)(x_n) = f(a) + g(a)$. Therefore $f + g$ is continuous at a.

   (b) We have that $\lim_{n\to\infty} |f(x)| = |f(a)| = |f|(a)$ and so $|f$ is continuous at a.

   (c) By the product rule for sequences we have that $\lim_{n\to\infty}(f \cdot g)(x_n) = f(a)g(a)$. Therefore $f \cdot g$ is continuous at a.

   (d) Since $g(a) \neq 0$ and $g(x_n) \neq 0$ for all $n \in \mathbb{N}$ by the quotient rule for sequences we have that

$$\lim_{n\to\infty} h(x_n) = \lim_{n\to\infty} \frac{f(x_n)}{g(x_n)} = \frac{a}{b}.$$

9. Let $x \in [0, \infty)$ and let (x_n) a sequence where each $x_n \in [0, \infty)$ and $\lim_{n\to\infty} x_n = x$. We have by exercise sheet 5 question 1 $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \sqrt{x_n} = \sqrt{x} = f(x)$. Thus f is continuous at x. So f is continuous on $[0, \infty)$.

10. We claim that f is continuous at a if and only if $a \notin \mathbb{Z}$. First of all suppose that $x \notin \mathbb{Z}$. Thus there exists $k \in \mathbb{Z}$ such that $k < x < k + 1$ and so $f(x) = k$. Let (x_n) be a sequence with $\lim_{n\to\infty} x_n = x$ and let $\varepsilon = \min\{x - k, k + 1 - x\} > 0$. Since $\lim_{n\to\infty} x_n = x$ we can find $N \in \mathbb{N}$ such that for all $n \geq N$ we have that $|x_n - x| \leq \varepsilon/2$ and so $x_n \in (k, k+1)$ and so $f(x_n) = k$. Therefore $\lim_{n\to\infty} f(x_n) = k = f(x)$. Thus f is continuous at x.

    Now suppose that $x \in \mathbb{Z}$. Then for all $n \in \mathbb{N}$ $f(x - \frac{1}{2n}) = x - 1 \neq f(x) = x$. Since $\lim_{n\to\infty} x - \frac{1}{2n} = x$ we can conclude that f is discontinuous at x.