Solutions to exercise sheet 9: Extremal Value Theorem and Intermediate Value Theorem

1. Let $(x_n)_{n\in\mathbb{N}}$ satisfy that $x_n \in [0,1]$ for all $n \in \mathbb{N}$. Thus $|x_n| \leq 1$ for all $n \in \mathbb{N}$ and so (x_n) is bounded. Therefore by the Bolzano-Weierstrass theorem we can find a subsequence of (x_n), (x_{n_k}) which is convergent. Let $x = \lim_{k\to\infty} x_{n_k}$ and note that $x \in [0,1]$. Since f is continuous $f(x) = \lim_{k\to\infty} f(x_{n_k})$ and therefore $(f(x_{n_k}))$ is convergent.

2. (a) First of all we will show that $\{f(x) : x \in (0,1)\}$ is bounded above. Let $\varepsilon = 1$ since $\lim_{x\to 0} f(x) = 0 = \lim_{x\to 1} f(x)$ we can find $\delta > 0$ such that for all $x \in (0,1)$ if $x \leq \delta$ then $f(x) \leq 1$ and if $x \geq 1-\delta$ then $f(x) \leq 1$. If $\delta \geq \frac{1}{2}$ then $f(x) \leq 1$ for all $x \in (0,1)$ and so f is bounded. If $\delta < \frac{1}{2}$ then consider f restricted to the interval $[\delta, 1-\delta]$. By the extremal value theorem f is bounded above on the interval $[\delta, 1-\delta]$ so there exists K such that for all $x \in [\delta, 1-\delta]$, $f(x) \leq K$. Therefore for all $x \in (0,1)$ we have that $f(x) \leq \max\{K, 1\}$ and so f is bounded above.

   Now let $\alpha = \sup\{f(x) : x \in (0,1)\}$. Since $f(x) > 0$ for all $x \in (0,1)$ we can conclude that $\alpha > 0$. Therefore since $\lim_{x\to 0} f(x) = 0 = \lim_{x\to 1} f(x)$ we can find $\delta > 0$ such that for all $x \in (0,1)$ if $x \leq \delta$ then $f(x) \leq \alpha/2$ and if $x \geq 1-\delta$ then $f(x) \leq \alpha/2$. Therefore

   $$\sup\{f(x) : x \in [\delta, 1-\delta]\} = \alpha$$

   and so since $f : [\delta, 1-\delta] \to \mathbb{R}$ is continuous by the extreme value theorem we can find $z \in [\delta, 1-\delta]$ such that

   $$f(z) = \alpha$$

   and so there exists $z \in (0,1)$ such that

   $$f(z) = \sup\{f(x) : x \in (0,1)\}.$$

   (b) We will show that $\inf\{f(x) : x \in (0,1)\} = 0$ and therefore since $f(x) > 0$ for all $x \in (0,1)$ there does not exist $z \in (0,1)$ with

   $$f(z) = \inf\{f(x) : x \in (0,1)\} = 0.$$

   Since $f(x) : x \in (0,1)\} \subseteq (0,\infty)$ we know that $\inf\{f(x) : x \in (0,1)\} \geq 0$. Now let $\varepsilon > 0$, since $\lim_{x\to 0} f(x) = 0$ there exists $\delta > 0$ such that $f(\delta) \leq \varepsilon$. Therefore $\inf\{f(x) : x \in (0,1)\} \leq \varepsilon$. Since this holds for all $\varepsilon > 0$ this means that

   $$\inf\{f(x) : x \in (0,1)\} = 0.$$

3. Since $\lim_{x\to\infty} f(x) = 0$ we can find $r \in (0,\infty)$ such that if $x \geq r$ then $|f(x) - 0| \leq \frac{1}{2}$ and so $f(x) \leq \frac{1}{2}$. Now since $f : [0,r] \to \mathbb{R}$ is continuous we know by the extremal value theorem it is bounded above and there exists $z \in [0,r]$ such that

$$f(z) = \sup\{f(x) : x \in [0,r]\}.$$

However since $f(0) = 1$ we must have that

$$f(z) = \sup\{f(x) : x \in [0,r]\} \geq 1.$$

Thus since $f(x) \leq 1$ for all $x \geq r$ we know that $f(z) \geq f(x)$ for all $x \in [0,\infty)$ and so

$$f(z) = \sup\{f(x) : x \in [0,\infty)\}.$$

The question asks us to find y such that

$$f(y) = \sup\{f(x) : x \in (0,\infty)\}.$$

However

$$\sup\{f(x) : x \in (0,\infty)\} = \sup\{f(x) : x \in [0,\infty)\}$$

since f is contionuous at 0.

It is not always true that there will exist $y \in [0,\infty)$ with

$$f(y) = \inf\{f(x) : x \in (0,\infty).$$

Consider the function $f : [0,\infty) \to \mathbb{R}$ given by $f(x) = 1 - \frac{x}{x+1}$. We have that $f(0) = 1$, $\lim_{x\to\infty} f(x) = 0$ and $f(x) > 0$ for all $x \in [0,\infty)$. Therefore

$$\inf\{f(x) : x \in (0,\infty) = 0$$

but there exists no $x \in [0,\infty)$ with $f(x) = 0$.

4. (a) Define $f : [0,1] \to \mathbb{R}$ by $f(x) = x^4 + x^3 + 3x - 4$ for all $x \in [0,1]$. f is continuous on $[0,1]$, $f(0) = -4$, $f(1) = 1$ and so $0 \in (f(0), f(1))$. Thus by the intermediate value theorem there exists $z \in [0,1]$ such that $f(z) = 0$ and so $z^4 + z^3 + 3z - 4 = 0$ for some $z \in [0,1]$.

   (b) Define $f : [-1,1] \to \mathbb{R}$ by $f(x) = 2x^4 + x^3 + 3x^2 - 2$ for all $x \in [-1.1]$. We have that $f(-1) = 2$ and $f(1) = 4$ so we cannot use the intermediate value theorem directly to find a 0. However $f(0) = -2$ and since f is continuous on $[-1,0]$ and $0 \in (f(0), f(-1))$ and thus by the intermediate value theorem there exists $z \in (-1,0)$ such that $f(z) = 0$. Moreover f is continuous on $[0,1]$ and $0 \in (f(0), f(1))$, thus by the intermediate value theorem there exists $y \in (0,1)$ such that $f(y) = 0$. Therefore $2x^4 + x^3 + 3x^2 - 2 = 0$ has at least two solutions in $[-1,1]$.

(c) Define $f : [0,1[\to \mathbb{R}$ by $f(x) = x^5 - 4x + 1$ for all $x \in [0,1]$. We have that f is continuous on $[0,1]$, $f(0) = 1$ and $f(1) = -2$. Thus by the intermediate value thoerem there exists $x \in (0,1)$ such that $f(x) = 0$. Thus this value of x will satisfy $x^5 - 4x + 1 = 0$ and so will also satisfy that $x^5 + 1 = 4x$.

5. Since f is continuous on $[0,1]$ we can use the extremal value theorem to say that f is bounded and there exists $y \in [0,1]$ such that

$$f(y) = \inf\{f(c) : c \in [0,1]\}.$$

Since $f(x) > 0$ for all $x \in [0,1]$ it follows that $f(y) > 0$. So if we let $a = f(y)$ then $f(x) \geq a > 0$ for all $x \in [0,1]$.

6. Note: It can be very helpful to draw a graph for this type of question. Consider the function $g : [0,1] \to \mathbb{R}$ defined by $g(x) = 3x - f(x)$. If $g(0) = 0$ then $f(0) = 0$ and if $g(1) = 0$ then $f(1) = 3$. Thus we can suppose that $g(0) \neq 0$ and $g(1) \neq 3$. $g(0) = -f(0) \leq 0$ and since we suppose that $g(0) \neq 0$ we have that $g(0) < 0$. $g(1) = 3 - f(x) \geq 0$ and so since $g(1) > 0$. Therefore we have that $g : [0,1] \to \mathbb{R}$ is continuous with $0 \in (g(0), g(1))$ thus by the intermediate value theorem we can find $z \in (0,1)$ such that $g(z) = 0$. Therefore $3z - f(z) = 0$ and so $f(z) = 3z$.

7. Since $\lim_{x\to\infty} f(x) = 0$ there exists $r_0 \in [0,\infty)$ such that for all $x \geq r_0$, $f(x) \leq 1$. We let $r = \max\{r_0, 2\}$ and define $g : [0,r] \to \mathbb{R}$ by $g(x) = x - f(x)$. g will be continuous on $[0,r]$, $g(0) = -1$ and $g(r) = r - f(r) \geq 2 - 1 = 1$. Thus $0 \in (g(0), g(r))$ and so by the intermediate value theorem there exists $z \in (0,r)$ with $g(z) = 0$ and so $z \in (0,\infty)$ with $f(z) = z$.

8. Let $\alpha = \inf\{f(x) : x \in [0,1]\}$ and $\beta = \sup\{f(x), x \in [0,1]\}$. By the extreme value theorem there exist $y, z \in [0,1]$ where

$$\alpha = f(y) \text{ and } \beta = f(z)\}.$$

If $\alpha = \beta$ then for all $x \in [0,1]$ we have that $f(x) = \alpha$ and we can conclude that

$$f([0,1]) = \{\alpha\} = [\alpha, \alpha].$$

If $\alpha < \beta$ then we know that $y \neq z$. First of all suppose that $y < z$. We then have that $f : [y,z] \to \mathbb{R}$ is continuous and so by the intermediate value theorem for all $a \in (\alpha, \beta)$ there exists $x \in (y,z)$ such that $f(x) = a$. Therefore

$$f([0,1]) \subseteq [\alpha, \beta].$$

However $f(x) \in [\alpha, \beta]$ for all $x \in [0,1]$ and so

$$f([0,1]) = [\alpha, \beta].$$

Now suppose that $y > z$. We then have that $f : [z, y] \to \mathbb{R}$ is continuous and so by the intermediate value theorem for all $a \in (\alpha, \beta)$ there exists $x \in (z, y)$ such that $f(x) = a$. Therefore

$$f([0, 1]) \subseteq [\alpha, \beta].$$

However $f(x) \in [\alpha, \beta]$ for all $x \in [0, 1]$ and so

$$f([0, 1]) = [\alpha, \beta].$$

9. We can suppose without loss of generality that $x < y$. We know that $f : [x, y] \to \mathbb{R}$ is continuous. Since $f(x)f(y) < 0$ we know that $f(x), f(y) \neq 0$ and if $f(x) > 0$ then $f(y) < 0$ and if $f(y) > 0$ then $f(x) < 0$. In both cases we can apply the intermediate value theorem to say that there will exist $z \in [x, y]$ such that $f(z) = 0$.