Siegel modular forms and Hecke operators in degree 2

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Abstract

In our earlier paper [7], we presented an algorithm for computing explicitly the coset representatives and the action of the Hecke operators on Siegel modular forms for arbitrary degree. The action was expressed in terms of the effect on the lattice-Fourier coefficients of the form (see the introduction for a definition of this term). This expression involved combinatorial terms that count local geometric-lattice substructures.

In this paper, we concentrate on degree two and make these actions completely explicit. As a corollary, we derive a number of identities and applications. For example, we show that for Hecke eigenforms, a certain average lattice-Fourier coefficient is completely determined (and explicitly given) by the Hecke eigenvalues and the (average) lattice-Fourier coefficient for a maximal lattice. As another corollary, we express the Koecher-Maaß series for such a form as a

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product of three zeta-functions; these are the spinor zeta-function, the standard zeta-function and a new zeta-function supported on (globally) maximal lattices. This expansion is new and the first to connect the standard zeta-function to the Koecher-Maaß series.

Contents

1 Introduction

Let F be a degree 2, weight k Siegel modular form of level N which is an eigenfunction for all the Hecke operators. That is, for each prime p,

$$F|T(p) = \lambda(p)\ F, \qquad F|\widetilde{T}_1(p^2) = \lambda_1(p^2)\ F$$

where here $T(p)$ and $\widetilde{T}_1(p^2)$ denote the operators described in [7]. As a Fourier series,

$$F(\tau) = \sum_{\Lambda} c(\Lambda)\, \mathrm{e}^*\{\Lambda\tau\}$$

where τ is in the Siegel upper half-space, Λ varies over all isometry classes of rank 2 lattices equipped with a positive, semi-definite quadratic form Q, and with T a matrix representing Q on Λ,

$$\mathrm{e}^*\{\Lambda\tau\} = \sum_{G \in O(\Lambda)\backslash \mathrm{GL}_2(\mathbb{Z})} \exp(\pi i Tr({}^tGTG\tau)). \tag{1.1}$$

Here $O(\Lambda)$ is the orthogonal group of Λ (see O'Meara [10] for the precise definitions and results on lattices and quadratic forms that we use in this paper). We view Λ as a representative of the isometry class in which it lies. (Actually, when k is odd, we need to equip each isometry class with an orientation, and in the above sum, $G \in O^+(\Lambda)\backslash SL_2(\mathbb{Z})$; cf. [7].) Note that $c(\Lambda) = 0$ unless Λ is even integral.

In the above expansion, $c(\Lambda) = c(T)$ where $c(T)$ is a Fourier coefficient of F for some matrix T representing the quadratic form Q on Λ. The G in the sum in (1.1) represents a change of basis for the quadratic form on Λ. We call $c(\Lambda)$ a *lattice-Fourier coefficient* of F.

We have multiple goals in this paper. First, from this viewpoint of lattice-Fourier coefficients, we give explicit expressions for the action of the Hecke operators in some average sense (see (2.5) and (2.6)). Second, these formulas lead to relations between (average) lattice-Fourier coefficients and Hecke-eigenvalues for F (see Lemmas 2.4 and 2.5). Third, from these relations we compute a generating function for each family of average lattice-Fourier coefficients (locally) in terms of the eigenvalues and a maximal lattice in the family (Propositions 2.2 and 2.3). Fourth, as a consequence, we prove a multiplicativity or local-global property for these average lattice-Fourier coefficients (Theorem 2.1) as well as give explicit formulas for the local coefficients in terms of the Satake-parameters (Proposition 2.4 and Section 2.4.2). Fifth, we compute a factorization of the Koecher-Maaß series for F in terms of three zeta-functions (Theorem 3.1). The first two are the spinor zeta-function (already known as a factor, Andrianov [1, p. 84ff]) and the standard zeta-function (cf. Andrianov [2]). These two zeta-functions are completely determined by the Hecke-eigenvalues (and show no explicit connection with the coefficients of F). The third is a complicated zeta-function whose coefficients depend on the eigenvalues and the average lattice-Fourier coefficients for maximal lattices. Finally, we use the explicit formulas to give optimal (but relative) bounds for the average lattice-Fourier coefficients, whenever the Ramanujan-Petersson conjecture holds (Theorem 4.1).

Remark 1.1 *Our results raise to the foreground fundamental questions concerning the extent to which the Hecke eigenvalues characterize a Siegel modular form of degree* 2 *(the so-called multiplicity one problem). All our theorems express the complete dependence between the eigenvalues and lattice-Fourier coefficients but only* relative to the coefficients for maximal lattices. *In other words, the eigenvalues and the maximal lattice coefficients seem to be completely independent. This is analogous to the half-integral weight case where there are limitations with respect to square-classes.*

This "information barrier" between eigenvalues and maximal lattice *coefficients raises these specific questions (among others):*

- *can there be a multiplicity-one theorem based solely on eigenvalues?*
- *can (optimal and absolute) bounds on Fourier coefficients be derived from Ramanujan-Petersson bounds on eigenvalues?*
- *are there other invariants (besides eigenvalues) which summarize the uniqueness of coefficients for maximal lattices?*

Our results suggest that the answer to the first two questions is "no". The answer to the third question is probably "yes", but we do not know what form such invariants must take.

We now begin by specifying some notation for lattices.

Let Λ be an integer lattice equipped with a even integral quadratic form Q. For a prime p, we let $\Lambda_{(p)}$ denote the local lattice $\mathbb{Z}_p \otimes \Lambda$. By Λ^u we mean the lattice "scaled" by u, that is, the lattice Λ equipped with the quadratic form uQ. By $u\Lambda$ we mean the lattice multiplied by u, that is, each vector is scaled by u. So $u\Lambda \simeq \Lambda^{u^2}$. Both of these scalings may occur in the global or local context.

Throughout, we will assume that our lattices satisfy $\operatorname{disc} \Lambda \neq 0$.

2 Hecke actions

Theorem 4.2 from [7] shows that, for p prime,

$$\lambda(p)c(\Lambda) = p^{2k-3}c(\Lambda^{1/p}) + p^{k-2} \sum_{\{\Lambda:\Omega\}=(1,p)} c(\Omega^{1/p}) + c(\Lambda^p). \tag{2.1}$$

Similarly, Theorem 4.1 of [7] gives us

$$\lambda_1(p^2)c(\Lambda) = p^{2k-3} \sum_{\{\Lambda:\Omega\}=(1/p,1)} c(\Omega) + p^{k-2}v_p(\Lambda)c(\Lambda) + \sum_{\{\Lambda:\Omega\}=(1,p)} c(\Omega) \tag{2.2}$$

where $v_p(\Lambda)$ is the number of isotropic lines in $\Lambda/p\Lambda$ (relative to the quadratic form $\frac{1}{2}Q$).

The above expansion is not completely specified until we compute the expression v_p and determine what Ω's can appear in the above summations. We do that in the next section, after incorporating some additional averaging.

Remark 2.1 *There are three operators which generate the local Hecke algebra:* $T(p)$ *and* $\widetilde{T}_1(p^2)$ *described above, and a third denoted by* $\widetilde{T}_2(p^2)$.

The third is algebraically dependent on the first two (see [7, Prop. 5.1]). Consequently, it provides no additional information about the relationship of coefficients and eigenvalues and is not used in this paper.

2.1 Average lattice-Fourier coefficients and Hecke action

Observe that these Hecke actions for a prime p carry a lattice-Fourier coefficient for a lattice Λ into a weighted sum of lattice-Fourier coefficients for some other lattices that agree locally with Λ at all places different from p and which have specific structural differences locally at p. These differences are relatively hard to describe globally without some averaging to "smooth" out the variances. We will do this averaging over isometry classes and over families, a notion which we define next.

Definition 2.1 $\Omega \in \operatorname{fam}\Lambda$ *if* Ω *and* Λ *have the same signature, and, at each finite prime,* Ω *looks like* Λ *scaled by a unit. That is,* $\Omega \in \operatorname{fam}\Lambda$ *if, for each finite prime* p*, there is a unit* u *in* $\mathbb{Z}_{(p)}$ *so that* $\Omega \simeq \Lambda^u$ *locally at* p*, and* $\mathbb{R}\otimes\Omega \simeq \mathbb{R}\otimes\Lambda$.

In the sum in (2.1) (and analogously the sums in (2.2), we sort the Ω into families and then into isometry classes. Thus

$$\sum_{\{\Lambda:\Omega\}=(1,p)} c(\Omega^{1/p}) = \sum_{\substack{\operatorname{fam}\Omega_0 \\ \operatorname{cls}\Omega\in\operatorname{fam}\Omega_0}} c(\Omega^{1/p})\#\{\Omega' \in \operatorname{cls}\Omega : \{\Lambda:\Omega'\} = (1,p)\}. \tag{2.3}$$

Given isometries σ, σ', we have $\sigma\Omega = \sigma'\Omega$ if and only if $\sigma^{-1}\sigma' \in O(\Omega)$, the orthogonal group of Ω. If $\Omega' \in \operatorname{cls}\Omega$ and $\{\Lambda : \Omega'\} = (1,p)$, then there exists an isometry σ such that $\{\Lambda : \sigma\Omega\} = (1,p)$. Thus, given such an Ω' there exist $o(\Omega)$ isometries σ such that $\Omega' = \sigma\Omega$. (Here $o(\Omega)$ denotes the order of the orthogonal group $O(\Omega)$. Since $\operatorname{disc}\Omega \neq 0$, we know the lattices on $\mathbb{Q}\Omega$ are positive definite, so their orthogonal groups are finite.) Hence, with σ denoting an isometry,

$$\#\{\Omega' \in \operatorname{cls}\Omega : \{\Lambda:\Omega'\} = (1,p)\} = \frac{\#\{\sigma :\ \{\Lambda:\sigma\Omega\} = (1,p)\}}{o(\Omega)}.$$

Consequently, we can average over families in (2.3) as follows:

$$\begin{aligned}
&\sum_{\operatorname{cls}\Lambda\in\operatorname{fam}\Lambda_0} \frac{1}{o(\Lambda)} \sum_{\{\Lambda:\Omega\}=(1,p)} c(\Omega^{1/p}) \\
&= \sum_{\operatorname{fam}\Omega_0} \sum_{\substack{\operatorname{cls}\Lambda\in\operatorname{fam}\Lambda_0 \\ \operatorname{cls}\Omega\in\operatorname{fam}\Omega_0}} \frac{c(\Omega^{1/p})}{o(\Omega)} \frac{\#\{\sigma :\ \{\Lambda:\sigma\Omega\} = (1,p)\}}{o(\Lambda)}
\end{aligned}$$

$$= \sum_{\mathrm{fam}\,\Omega_0} \sum_{\mathrm{cls}\,\Omega\in\mathrm{fam}\,\Omega_0} \frac{c(\Omega^{1/p})}{o(\Omega)} \sum_{\mathrm{cls}\,\Lambda\in\mathrm{fam}\,\Lambda_0} \frac{\#\{\sigma:\ \{\Omega:\sigma p\Lambda\}=(1,p)\}}{o(\Lambda)}$$

$$= \sum_{\substack{\mathrm{fam}\,\Omega_0\\ \mathrm{cls}\,\Omega\in\mathrm{fam}\,\Omega_0}} \frac{c(\Omega^{1/p})}{o(\Omega)} \#\{\Lambda'\in\mathrm{fam}\,\Lambda_0:\ \{\Omega:p\Lambda'\}=(1,p)\}$$

$$= \sum_{\substack{\mathrm{fam}\,\Omega_0\\ \mathrm{cls}\,\Omega\in\mathrm{fam}\,\Omega_0}} \frac{c(\Omega^{1/p})}{o(\Omega)} w_p(\Omega,p\Lambda), \tag{2.4}$$

where $w_p(\Lambda,\Omega)$ is the number of sublattices Ω' of Λ such that $\{\Lambda:\Omega'\} = (1,p)$ and such that $\Omega' \simeq \Omega^u$ for some p-unit u, that is,

$$w_p(\Lambda,\Omega) = \#\{\Omega'\in\mathrm{fam}\,\Omega:\ \{\Lambda:\Omega'\}=(1,p)\}.$$

It is trivial from the definition that w_p depends only on the family of either of its variables (i.e., is invariant under scaling by units in either variable). Explicit values for this w_p are given in Lemma 2.2.

These calculations suggest that we can simplify the expression of the Hecke actions (i.e., reduce the explicit calculation to computing $w_p(\Lambda,\Omega)$) by averaging over isometry classes (with the right weighting) and also over families. We need a couple of definitions.

Definition 2.2 *Define the* family mass *of a lattice as*

$$\mu(\Lambda_0) = \sum_{\mathrm{cls}\,\Lambda\in\mathrm{fam}\,\Lambda_0} \frac{1}{o(\Lambda)}$$

and define the average lattice-Fourier coefficient *of F by the expression*

$$a(\Lambda_0) = \frac{1}{\mu(\Lambda_0)} \sum_{\mathrm{cls}\,\Lambda\in\mathrm{fam}\,\Lambda_0} \frac{c(\Lambda)}{o(\Lambda)}.$$

Remark 2.2 *When F is in Maaß' Spezialchar, $a(\Lambda) = c(\Lambda)$. Consequently, in this case no averaging occurs and (most of) the results of the rest of this paper apply directly to the lattice-Fourier coefficients. It also follows that not all $a(\Lambda)$ are zero. Additionally (as we will see later), if the Koecher-Maaß series is not identically zero, then again not all $a(\Lambda)$ vanish.*

We will not be using the family mass for a while, but we will need a simple lemma immediately.

Lemma 2.1 *If $\{\Lambda:\Omega\} = (1,p)$, then*

$$\mu(\Lambda)\ w_p(\Lambda,\Omega) = \mu(\Omega)\ w_p(\Omega,p\Lambda).$$

Proof. The proof is easy from the definitions if we observe that $\#\{\sigma : \{\Omega : \sigma p\Lambda\} = (1,p)\} = \#\{\sigma : \{\Lambda : \sigma\Omega\} = (1,p)\}$. $\square$

With this lemma, the expression in (2.4) then equals

$$\sum_{\substack{\text{fam } \Omega_0 \\ \text{cls } \Omega \text{ fam } \Omega_0}} \frac{c(\Omega^{1/p})}{o(\Omega)} w_p(\Omega_0, p\Lambda_0) = \frac{\mu(\Lambda_0)}{\mu(\Omega_0)} \sum_{\substack{\text{fam } \Omega_0 \\ \text{cls } \Omega \text{ fam } \Omega_0}} \frac{c(\Omega^{1/p})}{o(\Omega)} w_p(\Lambda_0, \Omega_0).$$

Consequently, we have from the above, (2.4), (2.1), and (2.2)

$$\begin{aligned} \lambda(p)a(\Lambda) &= p^{2k-3}a(\Lambda^{1/p}) \\ &\quad + p^{k-2} \sum_{\text{fam } \Omega} w_p(\Lambda,\Omega)a(\Omega^{1/p}) + a(\Lambda^p) \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} \lambda_1(p^2)a(\Lambda) &= p^{2k-3} \sum_{\text{fam } \Omega} w_p(\Lambda,\Omega)a(\frac{1}{p}\Omega) \\ &\quad + p^{k-2}v_p(\Lambda)a(\Lambda) + \sum_{\text{fam } \Omega} w_p(\Lambda,\Omega)a(\Omega) \end{aligned} \tag{2.6}$$

These relations provide the most compact and explict expressions for the Hecke actions in degree 2.

We conclude this section by giving the explicit values for the combinatorial expressions $w_p(\Lambda,\Omega)$ and $v_p(\Lambda)$.

Lemma 2.2 *Fix a prime $p \not| N$ and suppose $\Lambda_{(q)} \simeq \Omega_{(q)}$ at each prime $q \neq p$. Let r and m be positive integers. We have*

$$w_p(\Lambda^{p^m}, \Omega^{p^m}) = w_p(\Lambda, \Omega).$$

For p odd, δ a non-square p-unit and ϵ an arbitrary p-unit, we have

$$w_p(\Lambda,\Omega) = \begin{cases} 2 & \text{if } \Lambda_{(p)} \simeq \langle 1,-1\rangle,\ \Omega_{(p)} \simeq p\langle 1,-1\rangle, \\ p-1 & \text{if } \Lambda_{(p)} \simeq \langle 1,-1\rangle,\ \Omega_{(p)} \simeq \langle 1\rangle \perp p^2\langle -1\rangle, \\ p+1 & \text{if } \Lambda_{(p)} \simeq \langle 1,-\delta\rangle,\ \Omega_{(p)} \simeq \langle 1\rangle \perp p^2\langle -\delta\rangle, \\ p & \text{if } \Lambda_{(p)} \simeq \langle 1\rangle \perp p^r\langle -\epsilon\rangle,\ \Omega_{(p)} \simeq \langle 1\rangle \perp p^{r+2}\langle -\epsilon\rangle, \\ 1 & \text{if } \Lambda_{(p)} \simeq \langle 1\rangle \perp p^r\langle -\epsilon\rangle,\ \Omega_{(p)} \simeq p^2\langle 1\rangle \perp p^r\langle -\epsilon\rangle, \\ 0 & \text{otherwise.} \end{cases}$$

For $p=2$ and ϵ an arbitrary 2-unit, we have

$$w_2(\Lambda,\Omega)=\begin{cases} 2 \ \textit{if}\ \Lambda_{(2)}\simeq\begin{pmatrix}0&1\\1&0\end{pmatrix},\ \Omega_{(2)}\simeq 2\begin{pmatrix}0&1\\1&0\end{pmatrix},\\ 1 \ \textit{if}\ \Lambda_{(2)}\simeq\begin{pmatrix}0&1\\1&0\end{pmatrix},\ \Omega_{(2)}\simeq\langle 2,-2\rangle,\\ 3 \ \textit{if}\ \Lambda_{(2)}\simeq\begin{pmatrix}2&1\\1&2\end{pmatrix},\ \Omega_{(2)}\simeq\langle 2,6\rangle,\\ 1 \ \textit{if}\ \Lambda_{(2)}\simeq\langle 2,-2\rangle,\ \Omega_{(2)}\simeq 2^2\begin{pmatrix}0&1\\1&0\end{pmatrix},\\ 1 \ \textit{if}\ \Lambda_{(2)}\simeq\langle 2,6\rangle,\ \Omega_{(2)}\simeq 2^2\begin{pmatrix}2&1\\1&2\end{pmatrix},\\ 1 \ \textit{if}\ \Lambda_{(2)}\simeq\langle 2,-2\epsilon\rangle,\ \Omega_{(2)}\simeq 2\langle 2,-2\epsilon\rangle,\ \epsilon=-1,3,\\ 2 \ \textit{if}\ \Lambda_{(2)}\simeq\langle 2,-2\epsilon\rangle,\ \Omega_{(2)}\simeq\langle 2\rangle\perp 2^2\langle -2\epsilon\rangle,\\ 2 \ \textit{if}\ \Lambda_{(2)}\simeq\langle 2\rangle\perp 2^r\langle -2\epsilon\rangle,\ \Omega_{(2)}\simeq\langle 2\rangle\perp 2^{r+2}\langle -2\epsilon\rangle,\\ 1 \ \textit{if}\ \Lambda_{(2)}\simeq\langle 2\rangle\perp 2^r\langle -2\epsilon\rangle,\ \Omega_{(2)}\simeq 2^2\langle 2\rangle\perp 2^r\langle -2\epsilon\rangle,\\ 0 \ \textit{otherwise.}\end{cases}$$

Remark 2.3 *Note that $w_p(\Lambda,\Omega)=0$ unless $\det(\Omega)=p^2\det(\Lambda)$. More precisely, either Ω is Λ scaled by p or one Jordan component is "shifted" by p^2. Additionally, no change is made to the square-class of the unit.*

Proof. For $p\neq 2$ and given $\Lambda_{(p)}$, we need to find all sublattices $\Omega_{(p)}$ with $\{\Lambda:\Omega\}=(1,p)$ (up to scaling by units). Such $\Omega_{(p)}$ are the preimages in $\Lambda_{(p)}$ of lines in the space $\Lambda_{(p)}/p\Lambda_{(p)}\simeq\Lambda/p\Lambda$. The reader is referred to O'Meara [10, Sections 91C, 92:1-2, 93B] for related background.

We do the cases where $\Lambda_{(p)}\simeq\langle 1,-1\rangle$ relative to some bases x,y and leave the rest to the reader as the computations are similar. In this case, the lines in $\Lambda/p\Lambda$ are generated by $\bar{x},\bar{y}$ and $\bar{x}+\beta\bar{y},\bar{y}$ for $\beta\not\equiv 0 \bmod p$. The first two cases give us $\Omega_{(p)}=\mathbb{Z}_p x\oplus\mathbb{Z}_p(py)$ and $\Omega_{(p)}=\mathbb{Z}_p(px)\oplus\mathbb{Z}_p y$. Both of these are equivalent to $p\langle 1,-1\rangle$ and this proves the first part of this case.

Now take $\beta\not\equiv 0 \bmod p$. Then $\Omega_{(p)}=\mathbb{Z}_p(x+\beta y)\oplus\mathbb{Z}_p(py)\simeq\begin{pmatrix}2\beta&p\\p&0\end{pmatrix}$. But the latter is equivalent to $\langle 2\beta\rangle\perp p^2\langle -2\beta\rangle$. Scaling by a unit, this is $\langle 1\rangle\perp p^2\langle -1\rangle$. Consequently, this occurs $p-1$ times. This completes the proof for this case. □

Lemma 2.3 *For p odd, a non-square p-unit δ, a p-unit ϵ and integer $r\geq 1$, we have*

$$v_p(\Lambda)=\begin{cases} 2 & \textit{if}\ \Lambda_{(p)}\simeq\langle 1,-1\rangle,\\ 0 & \textit{if}\ \Lambda_{(p)}\simeq\langle 1,-\delta\rangle,\\ 1 & \textit{if}\ \Lambda_{(p)}\simeq\langle 1\rangle\perp p^r\langle -\epsilon\rangle,\\ p+1 & \textit{otherwise.}\end{cases}$$

Similarly, for $p = 2$, ϵ a 2-unit and integer $r \geq 0$, we have

$$v_2(\Lambda) = \begin{cases} 2 \text{ if } \Lambda_{(2)} \simeq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ 0 \text{ if } \Lambda_{(2)} \simeq \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \\ 1 \text{ if } \Lambda_{(2)} \simeq \langle 2 \rangle \perp 2^r \langle -2\epsilon \rangle, \\ 3 \text{ otherwise.} \end{cases}$$

Proof. This is just a special case of Proposition 4.1 in [7]; see the remark following that result. □

2.2 Explicit Hecke actions and family types

For an odd prime p, a lattice family in degree 2 is determined by three parameters: (a) the square class of a local unit, (b) the *split* p-structure without scaling (this can also be referred to as the primitivity) and (c) the *scaling* by a power of p. This can be expressed by $\Lambda_{(p)} \simeq \Lambda_r^{p^m}$ where in Jordan decomposition $\Lambda_r = \Lambda_r(t, \epsilon) \simeq \langle 1 \rangle \perp p^{2r+t} \langle -\epsilon \rangle$, ϵ is the local unit, $t = 0, 1$ reflects the splitting in the maximal lattice, $2r + t$ reflects the total local splitting, r reflects (half of) the non-maximal extra splitting and m reflects the amount of local scaling.

Lemma 2.2 shows us that the Hecke action preserves the parameters (t, ϵ) and changes only r and m. That is, the lattices Ω such that $w_p(\Lambda, \Omega) \neq 0$ have the same (t, ϵ) values as Λ. (This explains the choice of parameters $2r + t$ for the split structure.) Consequently, we can partition the set of lattices according to these parameters. We call each partition a *family type*. There are three family types described by the terms hyperbolic, anisotropic and split (which characterize the maximal lattice Λ_0 in each family type). They can be parameterized via $\Lambda_0 = \Lambda_0(t, \epsilon)$ or via additional parameters u_0 and v_0 as follows:
Case 1 (*hyperbolic*): $t = 0$, $\epsilon = 1$ (a square), $u_0 = 2$, $v_0 = -1$;
Case 2 (*anisotropic*): $t = 0$, $\epsilon = \delta$ (a non-square), $u_0 = 0$, $v_0 = 1$;
Case 3 (*split*): $t = 1$, ϵ arbitrary, $u_0 = 1$, $v_0 = 0$,
where

$$\begin{aligned} u_0 &= u_0(\Lambda_0) = 1 + \left(\frac{-\operatorname{disc} \Lambda_0}{p} \right), \\ v_0 &= 1 - u_0, \end{aligned} \tag{2.7}$$

and the symbol $\left(\frac{*}{p}\right)$ is the Kronecker symbol.

For $p = 2$, there are four family types. One family type has a maximal lattice with Jordan decomposition $\Lambda_0 \simeq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, that is, unimodular hyperbolic. The other (unscaled) lattices in this family type are locally of the

form $\Lambda_r \simeq \langle 2\rangle \perp 2^{2r-2}\langle -2\rangle$ for $r \geq 1$. The second family type has maximal lattice with Jordan decomposition $\Lambda_0 \simeq \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, that is, unimodular, anisotropic, non-diagonal. The other (unscaled) lattices in this family are locally of the form $\Lambda_r \simeq \langle 2\rangle \perp 2^{2r-2}\langle 6\rangle$ for $r \geq 1$. The third family type contains the (unscaled) lattices of the form $\Lambda_r \simeq \langle 2\rangle \perp 2^{2r}\langle -2\epsilon\rangle$, for $r \geq 0$ and $\epsilon = -1, 3$. The maximal lattice is 2-modular, diagonal and non-split isotropic. Finally, the fourth family type contains (unscaled) lattices of the form $\Lambda_r \simeq \langle 2\rangle \perp 2^{2r+1}\langle -2\epsilon\rangle$, for $r \geq 0$ and ϵ arbitrary. The maximal lattice is split. With u_0 and v_0 as in (2.7), we can parameterize these family types as follows:
Case 1 (*hyperbolic*): $t = 0$, $\epsilon = 1$, $u_0 = 2$, $v_0 = -1$;
Case 2 (*anisotropic*): $t = 0$, $\epsilon = -3$, $u_0 = 1$, $v_0 = 0$;
Case 3a (*2-modular*): $t = 0$, $\epsilon = -1, 3$, $u_0 = 1$, $v_0 = 0$;
Case 3b (*split*): $t = 1$, ϵ arbitrary, $u_0 = 1$, $v_0 = 0$.
We will see that the latter two cases have combinatorially identical Hecke actions so that we combine these two into a single Case 3. Consequently, we can (loosely) refer to the three cases for $p = 2$ by the same terms and, conveniently, by the same parameter u_0 as for p odd.

In the following, we will use the notation

$$\Lambda_{(p)} \simeq \Lambda_r(u_0)^{p^m} \tag{2.8}$$

to indicate locally at p the family type (parameterized by u_0), the scaling (parameterized by m) and the extra splitting (parameterized by r).

Remark 2.4 *As noted, the parameter u_0 provides a simple device for parameterizing the family types. Additionally, $u_0 = v_p(\Lambda)$ if $\Lambda_{(p)}$ is maximal (see Lemma 2.3). Finally, v_0 seems extraneous; it is a notational convenience so that many formulas become easier to write down and to digest.*

We can now reformulate the Hecke actions in the form of recursion relations in the parameters r and m. Fix a prime p and assume that F is an eigenform for $T(p)$ and $\widetilde{T}_1(p^2)$, with eigenvalues $\lambda = \lambda(p)$ and $\lambda_1 = \lambda_1(p^2)$, respectively.

Since the Hecke action is local and depends only on the local family type, we define some notation which carries only the information relevant for the purposes at hand (though it suppresses a number of otherwise important data). For $\Lambda_{(p)} \simeq \Lambda_r^{p^m}$ with $r, m \geq 0$ (in the notation above for Λ_r), let

$$a(r,m) = a_p(r,m;u_0) = a(\Lambda), \tag{2.9}$$

where u_0 captures the family type of Λ at p and (r,m) set the extra scaling and component shift parameters at p. We will see later how this "local" notation relates globally.

Set $a(r,m)=0$ if r or m is negative.

For $r\geq 0, m\geq -1$, define the auxiliary sequence

$$b(r,m) \;=\; u_r a(r,m) + a(r-1,m+1) + (p+v_r)a(r+1,m-1)$$

where

$$u_r \;=\; \begin{cases} u_0 & \text{if } r=0 \\ 0 & \text{if } r\geq 1 \end{cases} \qquad \text{and} \qquad v_r \;=\; \begin{cases} v_0 & \text{if } r=0 \\ 0 & \text{if } r\geq 1. \end{cases}$$

Set $b(r,m)=0$ in all other cases. Note that $b(r,-1)=a(r-1,0)$ for $r\geq 0$.

Lemma 2.4 *Let p be a fixed prime not dividing the level N of F. The Hecke action of (2.5) is expressed for $r,m\geq 0$ by the relation*

$$\lambda a(r,m) = p^{2k-3}a(r,m-1) + p^{k-2}b(r,m) + a(r,m+1). \tag{2.10}$$

Additionally, the Hecke action of (2.6) is expressed for $r,m\geq 0$ by the relation

$$\lambda_1 a(r,m) = p^{2k-3}b(r,m-1) + p^{k-2}v(r,m)a(r,m) + b(r,m+1), \tag{2.11}$$

where

$$v(r,m) = \begin{cases} u_0 & \text{if } r=m=0 \\ 1 & \text{if } r\geq 1, m=0 \\ p+1 & \text{otherwise (i.e., } m\geq 1, r\geq 0\text{).} \end{cases}$$

Proof. The proof is essentially just a change of notation, together with Lemmas 2.2 and 2.3 and the observation that $v(r,m)=v_p(\Lambda)$. □

In the next lemma, we detail the action when $p|N$.

Lemma 2.5 *If $p|N$, then the Hecke action can be expressed by the relations*

$$\begin{aligned} \lambda a(r,m) \;&=\; a(r,m+1) \\ \lambda_1 a(r,m) \;&=\; b(r,m+1). \end{aligned} \tag{2.12}$$

Proof. The proof is immediate from (2.5) and (2.6). □

This next proposition is our first step towards relating the average lattice-Fourier coefficients for a lattice to those of a maximal lattice, globally.

Proposition 2.1 *If Δ is a globally maximal lattice such that $a(\Delta) = 0$, then $a(\Lambda) = 0$ for every lattice Λ contained in Δ.*

Proof. A straight-forward double-induction on the relations of Lemma 2.4 and 2.5 can be used to show that $a(0,0) = 0$ implies $a(r,m) = 0$ for all $r, m \geq 0$. But $a(0,0) = a_p(0,0;u_0) = a(\Delta)$ for every p, when Δ is globally maximal. The result follows immediately. □

Our next goal is to derive from the two-term recurrence relations in Lemmas 2.4 and 2.5 a generating function for $a(r,m)$. We do this in a series of lemmas in the next few subsections.

2.2.1 Some lemmas

By Proposition 2.1, we can assume without loss of generality that $a(0,0) = 1$. We revert to the explicit dependence on $a(0,0)$ in major statements of lemmas and propositions, for completeness. (Recall that $a(0,0)$ is an average lattice-Fourier coefficient for a lattice which is maximal at p, though the dependence on p (and the structure at other primes) is suppressed in the notation.)

For $m \in \mathbb{Z}$, let

$$A_m(x) = \sum_{r\geq 0} a(r,m)x^r$$
$$B_m(x) = \sum_{r\geq 0} b(r,m)x^r.$$

Lemma 2.6 *For any prime p, we have*

$$B_{-1}(x) = xA_0(x)$$
$$B_0(x) = xA_1(x) + u_0$$
$$B_1(x) = xA_2(x) + \frac{p}{x}A_0(x) - \frac{p}{x} + \beta$$

where

$$\beta = u_0 a(0,1) + v_0 a(1,0).$$

Proof. The first two formulas can be derived easily from the definitions, since $b(r,-1) = a(r-1,0)$ and $a(-1,0) = 0$ (for the first formula) and $b(r,0) = u_r a(r,0) + a(r-1,1)$ and $u_r = 0$ if $r \geq 1$ (for the second formula). The third formula is also straightforward but a bit more involved. We give the details. By the definition,

$$b(r,1) = u_r a(r,1) + a(r-1,2) + (p+v_r)a(r+1,0),$$

and $u_r = v_r = 0$ for $r \geq 1$. Consequenlty,

$$\begin{aligned} B_1(x) &= u_0 a(0,1) + v_0 a(1,0) + \sum_{r\geq 1} a(r-1,2)x^r + p\sum_{r\geq 0} a(r+1,0)x^r \\ &= u_0 a(0,1) + v_0 a(1,0) + xA_2(x) + \frac{p}{x}(A_0(x)-1), \end{aligned}$$

from which the result follows. □

Lemma 2.7 *If $p \nmid N$, we have*

$$\begin{aligned} a(0,1) &= (\lambda - u_0 p^{k-2}) \\ a(1,0) &= \frac{1}{p+v_0}((\lambda_1 - u_0 p^{k-2}) - u_0 a(0,1)). \end{aligned}$$

Proof. These are both derived from special cases of (2.10) and (2.11) with $r = m = 0$ and the definitions. □

Lemma 2.8 *Assume $p \nmid N$. For $m = 0, 1, 2$, there exist polynomials $N_m(x)$ (in x) of degree at most three such that*

$$A_m(x) = N_m(x)/G_0(x),$$

where

$$\begin{aligned} G_0(x) &= 1 - (\lambda_1/p^{k-1} - 1/p - 1)p^{k-2}x \\ &\quad + (\lambda^2/p^{2k-3} - 2\lambda_1/p^{k-1} + 2/p)p^{2k-4}x^2 \\ &\quad - (\lambda_1/p^{k-1} - 1/p - 1)p^{3k-6}x^3 + p^{4k-8}x^4. \end{aligned}$$

In particular,

$$N_0(x) = 1 + (p^{k-3}(p+v_0) - \beta/p)x + (\lambda u_0 + 2v_0 p^{k-2} - \beta)p^{k-3}x^2 + v_0 p^{3k-7}x^3.$$

Furthermore,

$$\lambda x N_0(x) + u_0 G_0(x) \equiv 0 \bmod (1 + xp^{k-2}).$$

Proof. For the proof, we will derive three linear equations in the "variables" $A_m(x)$, for $m = 0, 1, 2$ and then apply Cramer's rule. Take $m = 0$ in (2.10), multiply by x^r and sum on $r \geq 0$. This gives the first equation

$$\lambda A_0(x) - (1 + xp^{k-2})A_1(x) + 0 \cdot A_2(x) = u_0 p^{k-2}. \tag{2.13}$$

Next, take $m = 1$ in (2.10), multiply by x^{r+1} and sum on $r \geq 0$. This yields

$$\lambda x A_1(x) = p^{2k-3}xA_0(x) + p^{k-2}xB_1(x) + xA_2(x).$$

Now apply the third formula in Lemma 2.6 to get the second equation

$$p^{k-1}(1+xp^{k-2})A_0(x) - \lambda x A_1(x) + x(1+xp^{k-2})A_2(x) = p^{k-2}(p-\beta x).$$

Finally, take $m = 0$ in (2.11), multiply by x^{r+1} and sum on $r \geq 0$. This computation is a bit more involved. The immediate consequence of this step is the formula

$$\lambda_1 x A_0(x) = p^{2k-3} x B_{-1}(x) + p^{k-2} x \sum_{r\geq 0} v(r,0)a(r,0)x^r + xB_1(x).$$

The summation term here is

$$\sum_{r\geq 0} v(r,0)a(r,0)x^r = \sum_{r\geq 0} a(r,0)x^r - v_0 = A_0(x) - v_0.$$

Substitute this as well as the first and third formulas from Lemma 2.6 into the previous formula. This yields the third equation

$$(p-\lambda_1 x + xp^{k-2} + x^2 p^{2k-3})A_0(x) + 0 \cdot A_1(x) + x^2 A_2(x) = p + (v_0 p^{k-2} - \beta)x.$$

The result follows by solving the linear system for A_0, A_1, A_2. In particular, note that the coefficient matrix of the system is

$$\begin{pmatrix} \lambda & -(1+xp^{k-2}) & 0 \\ p^{k-1}(1+xp^{k-2}) & -\lambda x & x(1+xp^{k-2}) \\ (p-\lambda_1 x + xp^{k-2} + x^2 p^{2k-3}) & 0 & x^2 \end{pmatrix}.$$

We observe that the last column is divisible by x. Its determinant is a polynomial in x of degree five, but with zero constant term and linear term $-px$. Call this denominator $-pxG_0(x)$ (so $G_0(x)$ has constant term 1 and degree 4). The constants in the system of equations is the vector

$$\begin{pmatrix} u_0 p^{k-2} \\ p^{k-2}(p-\beta x) \\ p + (v_0 p^{k-2} - \beta)x \end{pmatrix}.$$

It is easy to see that the numerator determinants are of degree at most 4. Also, the determinants for the numerator of A_0 and A_1 are divisible by x (as seen from their last column). The numerator determinant of A_2 vanishes at $x = 0$, so this is also divisible by x. Consequently, $N_m(x)$ $(m = 0, 1, 2)$ are polynomials in x, of degree at most 3. ($N_m(x)$ is the respective numerator determinant divided by $-px$.)

Additionally, it is easy to see that $G_0(x) \equiv \lambda^2 x^2/p \bmod (1+xp^{k-2})$ and $N_0(x) \equiv \lambda x^2 u_0 p^{k-3} \bmod (1+xp^{k-2})$. So the last statement in the lemma also holds.

Finally, the explicit expression for $G_0(x)$ and $N_0(x)$ come from direct calculation. □

Define two generating functions:

$$A(x,y) = \sum_{r,m\geq 0} a(r,m)x^r y^m$$
$$B(x,y) = \sum_{r,m\geq 0} b(r,m-1)x^r y^m.$$

Note that we start the B series with second index equal to -1 because $b(r,-1) = a(r-1,0)$ which is not zero for $r \geq 1$.

The next two lemmas will contain linear relations between these two generating functions from which we can compute $A(x,y)$ directly.

Lemma 2.9 *If $p \nmid N$, we have*

$$(p^{2k-3}y^2 - \lambda y + 1)A(x,y) + p^{k-2}B(x,y) = (1 + xp^{k-2})A_0(x).$$

Proof. Multiply (2.10) by $x^r y^{m+1}$ and sum on $r,m \geq 0$. We get

$$\begin{aligned}\lambda y A(x,y) &= p^{2k-3} \sum_{r,m\geq 0} a(r,m-1)x^r y^{m+1} \\ &\quad + p^{k-2} \sum_{r,m\geq 0} b(r,m)x^r y^{m+1} \\ &\quad + \sum_{r,m\geq 0} a(r,m+1)x^r y^{m+1}.\end{aligned}$$

The first sum on the right is easily seen to equal $y^2 A(x,y)$. The second sum is $B(x,y) - B_{-1}(x) = B(x,y) - xA_0(x)$ by Lemma 2.6. The third sum is $A(x,y) - A_0(x)$. Collecting terms we prove the lemma. □

Lemma 2.10 *If $p \nmid N$, there exists a polynomial $N_0^*(x,y)$ of degree 4 in x and 2 in y so that*

$$(p^{k-2}(p+1) - \lambda_1)y^2 A(x,y) + (p^{2k-3}y^2 + 1)B(x,y) = \frac{N_0^*(x,y)}{G_0(x)}.$$

Furthermore, the coefficient of x^4 in $N_0^*(x,y)$ is $v_0 p^{3k-7}(1 + y^2 p^{2k-3})$.

Proof. Multiply (2.11) by $x^r y^{m+2}$ and sum on $r,m \geq 0$. We get

$$\begin{aligned}\lambda_1 y^2 A(x,y) &= p^{2k-3}y^2 B(x,y) \\ &\quad + p^{k-2}y^2 \sum_{r,m\geq 0} v(r,m)a(r,m)x^r y^m \\ &\quad + \sum_{r,m\geq 0} b(r,m+1)x^r y^{m+2}.\end{aligned}$$

The last sum is $B(x,y) - B_{-1}(x) - yB_0(x) = B(x,y) - (xA_0(x) + yxA_1(x) + u_0 y)$ by Lemma 2.6. The second sum is (since $v(r,m) = p+1$ for $m \geq 1$)

$$\begin{aligned}(p+1)A(x,y) + \sum_{r,m\geq 0} & (v(r,0) - p - 1)a(r,0)x^r \\ &= (p+1)A(x,y) - p\sum_{r\geq 1} a(r,0)x^r - (v_0 + p) \\ &= (p+1)A(x,y) - pA_0(x) - v_0.\end{aligned}$$

Equation (2.13) computed in the proof of Lemma 2.8 has

$$(1 + xp^{k-2})A_1(x) = \lambda A_0(x) - u_0 p^{k-2}.$$

Collecting terms, we derive the relation

$$\begin{aligned}&(p^{k-2}(p+1) - \lambda_1)y^2 A(x,y) + (p^{2k-3}y^2 + 1)B(x,y) \qquad (2.14)\\ &\quad = (p^{k-1}y^2 + x + \frac{xy\lambda}{1+xp^{k-2}})A_0(x) + v_0 p^{k-2}y^2 + \frac{u_0 y}{1+xp^{k-2}}.\end{aligned}$$

It remains to show that the right hand side of (2.14) has the form specified in the statement of the lemma. Collecting terms, we write (2.14) in the form

$$(p^{k-2}y^2 + x)A_0(x) + v_0 p^{k-2}y^2 + \frac{y}{1+xp^{k-2}}(x\lambda A_0(x) + u_0).$$

By Lemma 2.8, the last part of this expression is

$$\frac{y}{(1+xp^{k-2})G_0(x)}(x\lambda N_0(x) + G_0(x)u_0) = y\frac{N^*(x)}{G_0(x)}$$

for some polynomial $N^*(x)$ of degree at most 3. We conclude the proof by putting

$$N_0^*(x,y) = (p^{k-2}y^2 + x)N_0(x) + v_0 p^{k-2}y^2 G_0(x) + yN^*(x)$$

and observing that the coefficient of x^4 in this expression is

$$v_0 p^{3k-7} + v_0 y^2 p^{5k-10} = v_0 p^{3k-7}(1 + y^2 p^{2k-3})$$

which is derived from the leading coefficients of N_0 and G_0 in Lemma 2.8. □

2.2.2 The generating function, $p \nmid N$

We are now in a position to compute the generating function for $a(r,m)$.

Proposition 2.2 *If $p \not| N$, there exists a polynomial $N_p(x,y;u_0)$ of degree 3 in x and degree 2 in y such that*

$$\sum_{r,m\geq 0} \frac{a_p(r,m;u_0)}{p^{r(k-2)+m(k-3/2)}} x^r y^m = a_p(0,0;u_0) \frac{N_p(x,y;u_0)}{G_p(x)H_p(y)}$$

where with

$$\begin{aligned} \lambda_0 &= \lambda/p^{k-3/2} \\ \lambda_2 &= \lambda_1/p^{k-1} + 1 - 1/p \end{aligned}$$

we have

$$\begin{aligned} G_p(x) &= 1 + (2-\lambda_2)x + (2+\lambda_0^2 - 2\lambda_2)x^2 + (2-\lambda_2)x^3 + x^4 \\ H_p(y) &= 1 - \lambda_0 y + \lambda_2 y^2 - \lambda_0 y^3 + y^4. \end{aligned}$$

Remark 2.5 *This result was first proved by Zagier (private communication) with $k = 2$ in our Case 1 above.*

Remark 2.6 *Note that the denominator factors G_p and H_p depend only on the eigenvalues and not on the family type. That dependence appears explicitly in N_p and implicitly in $a(0,0)$.*

Proof. Without loss of generality, we assume that $a_p(0,0;u_0) = 1$. The left hand side of the main result in this proposition is just $A(x/p^{k-2}, y/p^{k-3/2})$, so our first goal is to find $A(x,y)$ from the two linear equations in Lemmas 2.9 and 2.10. The denominator determinant is

$$\begin{vmatrix} (p^{2k-3}y^2 - \lambda y + 1) & p^{k-2} \\ (p^{k-2}(p+1) - \lambda_1)y^2 & (p^{2k-3}y^2 + 1) \end{vmatrix} = \begin{vmatrix} (y'^2 - \lambda_0 y' + 1) & p^{k-2} \\ p^{-k+2}(2-\lambda_2)y'^2 & (y'^2 + 1) \end{vmatrix}$$

where $y' = yp^{k-3/2}$. This is easily seen to equal $H_p(y') = H_p(yp^{k-3/2})$ by a computation.

The numerator determinant for $A(x,y)$ is

$$\begin{vmatrix} (1+xp^{k-2})N_0(x)/G_0(x) & p^{k-2} \\ N_0^*(x,y)/G_0(x) & (p^{2k-3}y^2+1) \end{vmatrix} = \frac{1}{G_0(x)} \begin{vmatrix} (1+xp^{k-2})N_0(x) & p^{k-2} \\ N_0^*(x,y) & (p^{2k-3}y^2+1) \end{vmatrix}.$$

The last determinant is a polynomial in x and y of degree 2 in y and at most degree 4 in x. But, the coefficient of x^4 in this expression is zero (using Lemmas 2.8 and 2.10), so that the actual degree is at most 3.

Observing that $G_p(x) = G_0(xp^{k-2})$, we complete the proof. □

Remark 2.7 *The specific form of the polynomial $N_p(x, y; u_0)$ in the proposition is not very important (mostly because it is so complicated). However, we give it here for completeness:*

$$
\begin{aligned}
N_p(x, y; u_0) = 1 &- \left[\frac{u_0}{p^{1/2}}\right] y - \left[\frac{v_0}{p}\right] y^2 \\
&+ \left[\frac{(u_0^2 + 3v_0 - v_0\lambda_2) - u_0\lambda_0 p^{1/2} + p}{p + v_0}\right] x \\
&+ \left[\frac{u_0(\lambda_2 - 1) - \lambda_0 p^{1/2}}{p^{1/2}}\right] xy \\
&+ \left[\frac{v_0(\lambda_2 - 2) + p}{p}\right] xy^2 \\
&+ \left[\frac{v_0^2 + u_0 v_0 \lambda_0 p^{1/2} + (u_0^2 + 3v_0 - v_0\lambda_2)p}{p(p + v_0)}\right] x^2 \\
&+ \left[\frac{u_0 v_0(\lambda_2 - \lambda_0^2 - 1) + (v_0\lambda_2 - u_0^2 - 2v_0)\lambda_0 p^{1/2} + u_0(\lambda_2 - 1)p}{p^{1/2}(p + v_0)}\right] x^2 y \\
&+ \left[\frac{v_0^2(2\lambda_2 - \lambda_0^2 - 2) + (u_0^2 + v_0(1 + \lambda_2 - \lambda_0^2))p - u_0\lambda_0 p^{3/2} + p^2}{p(p + v_0)}\right] x^2 y^2 \\
&+ \left[\frac{v_0}{p}\right] x^3 \\
&+ \left[\frac{-(v_0\lambda_0 + u_0 p^{1/2})}{p}\right] x^3 y \\
&+ \left[\frac{v_0^2(\lambda_2 - 1) + u_0 v_0 \lambda_0 p^{1/2} + (u_0^2 + v_0)p}{p(p + v_0)}\right] x^3 y^2.
\end{aligned}
$$

In none of the cases (that is, for particular choices of u_0 and $v_0 = 1 - u_0$) does it appear that this expression gets significantly simpler. For example, it does not factor (symbolically). However, $N_p(0, p^{1/2}y; u_0)$ has very simple forms in each case: $(1 - y)^2$, $1 - y^2$, and $1 - y$, respectively. On the other hand, $N_p(x, 0; u_0)$ is cubic, irreducible in the hyperbolic case, a product of a linear and quadratic in the anisotropic case and is (curiously) only a quadratic in the split case.

2.2.3 The generating function, $p|N$

In this section, we use the two-term recurrence relations in Lemma 2.5 to derive a generating function for the sequence $a(r, m)$ in this special case.

It is easy to see that $a(r,m) = \lambda^m a(r,0)$ so that

$$A(x,y) = \frac{A_0(x)}{(1-\lambda y)}.$$

It remains to compute $A_0(x)$. From (2.12) with $m = 0$ and Lemma 2.6, we get

$$\lambda_1 A_0(x) = B_1(x) = xA_2(x) + \frac{p}{x}A_0(x) - \frac{p}{x} + \beta.$$

But $A_2(x) = \lambda^2 A_0(x)$, so

$$A_0(x) = \frac{p - \beta x}{p - \lambda_1 x + \lambda^2 x^2}.$$

Also, $\beta = u_0 a(0,1) + v_0 a(1,0)$ and $a(0,1) = \lambda a(0,0) = \lambda$ and

$$a(1,0) = \frac{\lambda_1 - u_0\lambda}{p + v_0},$$

which is easily seen from (2.12) with $r = m = 0$. Thus, reinserting the explicit dependence on $a(0,0)$,

$$A(x,y) = a(0,0)\frac{p(p+v_0) - x(p\lambda u_0 + v_0\lambda_1)}{(p+v_0)(p - \lambda_1 x + \lambda^2 x^2)(1-\lambda y)}. \qquad (2.15)$$

We reformulate this in the next proposition in a form which is notationally consistent with the case $p \nmid N$.

Proposition 2.3 *If $p|N$, we have*

$$\sum_{r,m\geq 0} \frac{a_p(r,m;u_0)}{p^{r(k-2)+m(k-3/2)}} x^r y^m = a_p(0,0;u_0)\frac{N_p(x,y;u_0)}{G_p(x)H_p(y)},$$

where with

$$\begin{aligned} \lambda_0 &= \lambda/p^{k-3/2} \\ \lambda_2 &= \lambda_1/p^{k-1} + 1 - 1/p \end{aligned}$$

we have

$$\begin{aligned} N_p(x,y;u_0) &= 1 - \left[\frac{(\lambda_2 - 1 + 1/p)v_0 + \lambda_0\sqrt{p}u_0}{(p+v_0)}\right] x \\ G_p(x) &= 1 - (\lambda_2 - 1 + 1/p)x + \lambda_0^2 x^2 \\ H_p(y) &= 1 - \lambda_0 y. \end{aligned}$$

Note that, in contrast to the formula in Proposition 2.2, the numerator $N_p(x,y;u_0)$ is independent of y. Additionally, the factor $G_p(x)$ is quadratic even though the factor $H_p(y)$ is linear and neither is monic (as a side-effect of our uniform normalization in x and y). The proof of this proposition is immediate from (2.15).

2.3 A local-global property

In this section, we prove a local-global property for the average lattice-Fourier coefficients of Hecke eigenforms. This theorem also shows that the average lattice-Fourier coefficients for a Hecke eigenform are completely determined by the coefficients on maximal lattices and on the eigenvalues. This is a weak form of a multiplicity one theorem and should be compared with results of Breulmann-Kohnen [4], the recent result of Scharlau-Walling [14] and others. We will see other applications of this notion later.

First we introduce additional notation. In the expansions of the expressions $N_p(x,y;u_0)/(G_p(x)H_p(y))$ in Propositions 2.2 and 2.3, let $\lambda_p(r,m;u_0)$ be the coefficient of $x^r y^m$, so that

$$\sum_{r,m\geq 0} \lambda_p(r,m;u_0)x^r y^m = \frac{N_p(x,y;u_0)}{G_p(x)H_p(y)}, \tag{2.16}$$

or

$$\lambda_p(r,m;u_0) = \frac{a_p(r,m;u_0)/a_p(0,0;u_0)}{p^{r(k-2)+m(k-3/2)}}$$

(when $a_p(0,0;u_0) \neq 0$).

Theorem 2.1 *Let Λ be a (global) lattice and let Δ be a maximal lattice containing Λ. Then*

$$a(\Lambda) = a(\Delta)\prod_p \lambda_p(r_p,m_p;u_0)p^{r_p(k-2)+m_p(k-3/2)}. \tag{2.17}$$

Here the triple (r_p,m_p,u_0) is determined by $\Lambda_{(p)} \simeq \Lambda_{r_p}(u_0)^{p^{m_p}}$, in the notation of (2.8).

Remark 2.8 *The product in the above expression is actually finite. The only factors that are non-trivial (i.e., not equal to one) are for those primes p at which Λ is not maximal.*

Remark 2.9 *This theorem provides a refinement of Proposition 2.1.*

Proof. The proof is straightforward by induction on the number of primes at which Λ is not maximal. Assume Λ is maximal, then the theorem holds as $\lambda_p(0,0;u_0) = 1$ for all p. Assume Λ is not maximal at n primes. Then pick any such prime p. We use the results of Propositions 2.2 or 2.3 (depending on whether $p \not| N$ or $p|N$, respectively) to express

$$a(\Lambda) = a(\Lambda_p)\lambda_p(r_p,m_p;u_0)p^{r_p(k-2)+m_p(k-3/2)} \tag{2.18}$$

where (in the notation of those propositions) $a(\Lambda_p) = a_p(0, 0; u_0)$ for Λ_p isometric to Λ at all primes not equal to p and maximal at p and $\lambda_p(r_p, m_p; u_0)$ and u_0 are defined in (2.16) and (2.7), respectively. Now Λ_p has $n - 1$ primes at which it is not maximal. By the induction hypothesis, this is can be factored as in the statement of the theorem. This completes the proof. □

Remark 2.10 *As seen in the proof, the expressions $\lambda_p(r_p, m_p; u_0)$ are, by Propositions 2.2 and 2.3, polynomials in the eigenvalues $\lambda(p)$ and $\lambda_1(p^2)$. The explicit formulation depends on the family type (as expressed by the parameter u_0) and by the extra splitting and scaling (r_p, m_p). This justifies the statements made prior to the theorem.*

2.4 Explicit formulas via partial fractions

With the explicit generating functions given above, we can use the theory of partial fractions to derive explicit formulas for the coefficients $\lambda_p(r, m; u_0)$ of the generating functions. These explicit formulas will be stated in terms of the roots of the polynomial $H_p(y)$. We will see that these roots are connected to the Satake parameters (Satake [13]). (This is clear because of the relationship to the spinor zeta-function, but we make the connection explicit below.) However, the formulation of the results seem to be easiest and cleanest to state, not directly in terms of the Satake parameters, but in terms of these roots. In the next section we deal with the case where $p \nmid N$. We follow that with the case where $p|N$.

2.4.1 Expansion for $p \nmid N$

Both $H_p(y)$ and $G_p(x)$ are symmetric polynomials (that is, $y^4 H_p(1/y) = H_p(y)$). This means that if a is a root, then so is $1/a$. Let $a_1, a_2, 1/a_1, 1/a_2$ be the roots of $H_p(y)$; we can then express λ_0 and λ_2 as rational functions in a_1, a_2:

$$\begin{aligned}\lambda_0 &= a_1 + a_2 + 1/a_1 + 1/a_2,\\ \lambda_2 &= a_1 a_2 + a_1/a_2 + a_2/a_1 + 1/(a_1 a_2) + 2.\end{aligned}$$

We find by simple algebra that the roots of $G_p(x)$ are $a_1 a_2, a_1/a_2, 1/(a_1 a_2)$ and a_2/a_1. In other words,

$$\begin{aligned}H_p(y) &= (1 - y/a_1)(1 - y/a_2)(1 - a_1 y)(1 - a_2 y)\\ G_p(x) &= (1 - x/(a_1 a_2))(1 - a_2 x/a_1)(1 - a_1 a_2 x)(1 - a_1 x/a_2).\end{aligned}$$

By the same token, we can express $N_p(x, y; u_0)$ strictly in terms of these parameters and $q = p^{1/2}$ (but we don't write out that expansion here).

We've chosen the notation for the roots in a convenient way for what follows. However, it turns out that the a_1, a_2 are related to the Satake parameters because H_p is essentially the local factor of the Spinor zeta-function (Satake [13], Shimura [15] and Andrianov [1, p. 66ff] and the references therein). If $\alpha_0, \alpha_1, \alpha_2$ are the Satake parameters, then one formulation of the relationship is $\alpha_0 = (1/a_1)p^{k-3/2}, \alpha_1 = a_1/a_2, \alpha_2 = a_1 a_2$. This means that $G_p(x) = (1-\alpha_1 x)(1-\alpha_2 x)(1-x/\alpha_1)(1-x/\alpha_2)$. It follows that G_p is exactly the local factor of the standard zeta-function (Andrianov [2]).

For $p|N$, H_p is linear and G_p can be factored as $G_p(x) = (1-\lambda_0 b_1 x)(1-\lambda_0 x/b_1)$ where $b_1 + 1/b_1 = (\lambda_2 - 1 + 1/p)/\lambda_0$. Again there are two parameters defining the roots of G_p and H_p, namely, b_1 and λ_0.

The more interesting fact now is that we can expand the quotient $N_p/(G_p \cdot H_p)$ in a partial fraction decomposition with denominators consisting of one linear factor from G_p and one from H_p, with constant coefficients (for fixed a_1, a_2, p). Begin with the following:

Genericity Condition: *For all primes p, all of the roots of G_p or H_p are simple. Equivalently, for $p \not| N$, $a_1 \neq \pm a_2^{\pm 1}$ and $a_1, a_2 \neq \pm 1$. For $p|N$, $b_1 \neq \pm 1$.*

Remark 2.11 *This condition is only a technical convenience in order to avoid excessive cases and notation. It is possible to perform all the necessary calculations in the cases where the assumption is false; we leave that to the reader, however.*

Now, we set some additional notation by setting

$$\begin{aligned} q &= \sqrt{p} \\ P(q,u,v;u_0) &= (q-u)(q-v)(q+v_0 u)(q+v_0 v) \\ U(x,y,a_1,a_2) &= (1-a_1 a_2 x)(1-a_1 y) \\ D(a_1,a_2) &= (a_1 a_2)^{-2}(1-a_1^2)(1-a_2^2)(1-a_1 a_2)(a_2-a_1) \\ &= (a_1 - \frac{1}{a_1})(a_2 - \frac{1}{a_2})(\sqrt{\frac{a_1}{a_2}} - \sqrt{\frac{a_2}{a_1}})(\sqrt{a_1 a_2} - \frac{1}{\sqrt{a_1 a_2}}). \end{aligned}$$

Recall that $u_0 = 1 - v_0$ and that $\lambda_p(r,m;u_0)$ is the coefficient of $x^r y^m$ in $N_p(x,y;u_0)/(G_p(x)H_p(y))$.

Let W_0 be the group of actions on a_1, a_2 defined by $W_0 = \langle\sigma_0\rangle \times \langle\sigma_1\rangle \times \langle\sigma_2\rangle$ where σ_0 permutes a_1 and a_2, and σ_i $(i = 1,2)$ inverts a_i. (This is actually a part of the Weyl group when viewed as acting on the Satake parameters.) Then, the partial fraction expansion of the generating function is given by

$$\frac{N_p(x,y;u_0)}{G_p(x)H_p(y)}$$

$$= \frac{1}{D(a_1,a_2)q^2(q^2+v_0)} \sum_{\sigma\in W_0} \operatorname{sgn}(\sigma) \left(\frac{a_1^2 a_2 P(q, a_1^{-1}, a_2^{-1}; u_0)}{U(x,y,a_1,a_2)} \right)^{\sigma}.$$

Remark 2.12 *In general, a partial fraction expansion with denominator $G_p(x)H_p(y)$ has 16 terms (four from G_p times four from H_p). However, with the particular numerator $N_p(x,y;u_0)$ we have, only eight terms survive. Although it is not entirely clear why this should be the case (computationally), it seems that it must be the case because W_0 has order eight.*

Note that with the exception of the leading factor depending on v_0, only the expression $P(q,u,v;u_0)$ depends on the family type. For completeness, we write down $P(q,u,v;u_0)$ for each family type since they are quite simple on specialization:
Case 1 (*hyperbolic*): $P(q,u,v;2) = (q-u)^2(q-v)^2$.
Case 2 (*anisotropic*): $P(q,u,v;0) = (q^2-u^2)(q^2-v^2)$.
Case 3 (*split*): $P(q,u,v;1) = q^2(q-u)(q-v)$.
This expansion into partial fractions immediately implies the following formula under the Genericity Condition.

$$\lambda_p(r,m;u_0) \tag{2.19}$$
$$= \frac{1}{D(a_1,a_2)q^2(q^2+v_0)} \sum_{\sigma\in W_0} \operatorname{sgn}(\sigma) \big(P(q, a_1^{-1}, a_2^{-1}; u_0)\ a_1^{r+m+2} a_2^{r+1} \big)^{\sigma}.$$

This formulation is still in not the most convenient form. We next want to interchange the summation on σ with the terms in $P(q,u,v;u_0)$. To this end, we let

$$\begin{aligned} \rho(r,m) &= \sum_{\sigma\in W_0} \operatorname{sgn}(\sigma)(a_1^{r+m+2} a_2^{r+1})^{\sigma} \\ &= (a_1^{m+r+2}a_2^{r+1} - a_1^{r+1}a_2^{m+r+2}) \\ &\quad - (a_1^{m+r+2}a_2^{-r-1} - a_1^{r+1}a_2^{-m-r-2}) \\ &\quad - (a_1^{-m-r-2}a_2^{r+1} - a_1^{-r-1}a_2^{m+r+2}) \\ &\quad + (a_1^{-m-r-2}a_2^{-r-1} - a_1^{-r-1}a_2^{-m-r-2}). \end{aligned} \tag{2.20}$$

This does not depend on the initial conditions, but is strictly a function of the parameters a_1, a_2.

Expanding $P(q,u,v;u_0)$ and collecting terms involving $\rho(r,m)$, we easily see that

$$\sum_{\sigma\in W_0} \operatorname{sgn}(\sigma)\big(P(q, a_1^{-1}, a_2^{-1}; u_0)\ a_1^{r+m+2}a_2^{r+1}\big)^{\sigma}$$
$$= P(q,U,V;u_0) \circ \rho(r,m)$$

$$
\begin{aligned}
= \ & q^2\big[q^2\rho(r,m) - u_0 q\rho(r,m-1) - v_0\rho(r,m-2)\big] \qquad (2.21)\\
& -u_0 q\big[q^2\rho(r-1,m+1) - u_0 q\rho(r-1,m) - v_0\rho(r-1,m-1)\big]\\
& -v_0\big[q^2\rho(r-2,m+2) - u_0 q\rho(r-2,m+1) + v_0\rho(r-2,m)\big],
\end{aligned}
$$

where U and V are the operators defined by

$$U \circ \rho(r,m) = \rho(r,m-1), \quad V \circ \rho(r,m) = \rho(r-1,m+1).$$

There is an interesting pattern in this expansion. The first parameter in ρ decreases and the second increases as one moves down the rows. Within one of these rows, only the second parameter changes (decreases). Additionally, the coefficients within each of these rows are $q^2, -u_0 q, -v_0$ (which are the coefficients of $N_p(0,y;u_0)$). Similarly, the coefficients between these rows are $q^2, -u_0 q, -v_0$. However, we failed to find any direct connection with these coefficients and $N_p(x,y;u_0)$, in general. Note that in Case 2, the second row and second columns do not appear. In Case 3, the third row and third column vanish.

We collect what we have so far in a proposition.

Proposition 2.4 *Assume the Genericity Condition. Then if $p \nmid N$,*

$$\lambda_p(r,m;u_0) = \frac{1}{D(a_1,a_2)q^2(q^2+v_0)} P(q,U,V;u_0) \circ \rho(r,m)$$

Remark 2.13 *The formula in Proposition 2.4 is the degree 2 generalization of the classical formula from degree 1 for Ramanujan's τ-function:*

$$\frac{\tau(p^m)}{p^{11m/2}} = \frac{a^{m+1} - a^{-m-1}}{a - a^{-1}}.$$

In our case, the denominator role is played by $D(a_1,a_2)$ (essentially). In this classical case, it is possible to explicitly divide the numerator by the denominator and therefore write $\tau(p^m)$ as a sum of powers of a^2. Furthermore, there is a formulation for $\tau(p^m)$ involving an angle θ, where $\tau(p)p^{-11/2} = 2\cos\theta$ which facilitates bounds on $\tau(p^m)$, particularly since $|a| = 1$. In Section 4, we do the analogous "division" in the degree 2 case, and provide the analogous bound (though we do not given the representation in terms of angular parameters).

2.4.2 Expansion for $p|N$

In Lemma 2.3), write

$$G_p(x) = (1 - \lambda_0 b_1 x)(1 - \lambda_0 x/b_1),$$

so that

$$b_1 + \frac{1}{b_1} = \frac{\lambda_2 - 1 + 1/p}{\lambda_0}.$$

Assume $b_1 \neq \pm 1$ (this is the Genericity Condition for this case). We can now write (after a moderately lengthy calculation)

$$\lambda_p(r, m; u_0) = \frac{\lambda_0^{m+r}}{D(b_1)(p + v_0)} \sum_{\sigma \in W_0} \mathrm{sgn}(\sigma)(P(q, b_1^{-1}; u_0)\ b_1^{r+1})^\sigma,$$

where W_0 is the group of two elements which acts on b_1 by inverse, and

$$\begin{aligned} q &= \sqrt{p}, \\ D(b_1) &= b_1^{-1}(b_1^2 - 1) = b_1 - 1/b_1, \\ P(q, u; u_0) &= q^2 - u_0 u q - v_0 u^2 = (q - u)(q + v_0 u). \end{aligned}$$

Set

$$\rho(r) = b_1^{r+1} - b_1^{-r-1} \tag{2.22}$$

and we find that

$$\lambda_p(r, m; u_0) = \frac{\lambda_0^{m+r}}{D(b_1)(p + v_0)}(q^2 \rho(r) - u_0 q \rho(r-1) - v_0 \rho(r-2)).$$

Note that in the three family type cases, $P(q, u; u_0) = (q - u)^2$ (hyperbolic), $q^2 - u^2$ (anisotropic) and $q(q - u)$ (split). Additionally, note the explicit dependence on λ_0. These two remarks should be viewed also in the context of the observations preceding (2.19) and the statement of Proposition 2.4.

3 Koecher-Maaß zeta-functions

The results above, in particular the explicit generating functions (Propositions 2.2 and 2.3) and relationship between a lattice and its maximal lattice (Theorem 2.1), allow us to express the Koecher-Maaß zeta-function for a Hecke-eigenform in a very special way. This is given in the next theorem.

Let $K_F(s)$ be the Koecher-Maaß zeta-function defined by

$$K_F(s) = \sum_{T > 0} \frac{c(T)}{\#\,\mathrm{aut}(T)} (\det T)^{-s},$$

where $c(T)$ are the Fourier coefficients of F (as functions of matrices). Collecting terms by summing over lattices, we can write this as

$$K_F(s) = \sum_{\Lambda} a(\Lambda) \mu(\Lambda) (\det \Lambda)^{-s},$$

where $a(\Lambda)$ is the average lattice-Fourier coefficient and $\mu(\Lambda)$ is the family mass as defined in Definition 2.2.

Theorem 3.1 *Let F be a Hecke eigenform of degree 2, weight k and level N. Then*

$$K_F(s) = Z_F(2s)Z_F^{\times}(2s) \sum_{\Delta \ \max} a(\Delta)\mu(\Delta)(\det \Delta)^{-s} N_\Delta(2s)$$

where the sum is over maximal lattices and

$$\begin{aligned} Z_F(s) &= \prod_p H_p(p^{k-3/2-s})^{-1}, \\ Z_F^{\times}(s) &= \prod_p G_p(p^{k-1-s})^{-1}, \\ N_\Delta(s) &= \prod_p N_p(p^{-s};\Delta), \\ N_p(x;\Delta) &= \left(1 + \frac{v_0}{p}\right) N_p(xp^{k-1}, xp^{k-3/2}; u_0) \\ &\quad - \left(\frac{v_0}{p}\right) N_p(0, xp^{k-3/2}; u_0) G_p(xp^{k-1}). \end{aligned}$$

and furthermore $H_p(y)$, $G_p(x)$ and $N_p(x, y; u_0)$ are the expressions in the generating functions for the family type of $\Delta_{(p)}$ (parameterized by u_0; see Propositions 2.2 and 2.3).

Proof. Write $K_F(s)$ as

$$\sum_{\Delta \ \max} a(\Delta)\mu(\Delta)(\det \Delta)^{-s} \sum_{\Lambda \subseteq \Delta} \frac{a(\Lambda)}{a(\Delta)} \frac{\mu(\Lambda)}{\mu(\Delta)} \left(\frac{\det \Lambda}{\det \Delta}\right)^{-s}. \tag{3.1}$$

That is, sum first on maximal lattices and then on all lattices contained within the maximal lattice. Note that in the inner sum, we can assume $a(\Delta) \neq 0$ as otherwise that term is missing from the expression (this is a consequence of Proposition 2.1).

We claim that the inner sum in (3.1) equals

$$\prod_p \sum_{r,m \geq 0} \lambda_p(r, m; u_0)\mu_p(p^r; v_0) p^{r(k-2-2s)+m(k-3/2-2s)}. \tag{3.2}$$

where $u_0 = 1 - v_0$ parameterizes the family type of $\Delta_{(p)}$, λ_p is defined in (2.16), and μ_p is defined by $\mu_p(p^r; u_0) = (p + v_0)p^{r-1}$ for $r \geq 1$ and $\mu_p(1) = 1$.

To justify (3.2), we proceed as follows. For a prime p, write $\Lambda_{(p)} \simeq \Lambda_r(u_0)^{p^m}$. Then $\Delta_{(p)} \simeq \Lambda_0(u_0)$. Let $l = \prod_p p^r$ and $n = \prod_p p^m$. These two integers give us a convenient parameterization of Λ.

Given this parameterization, Theorem 2.1 gives us

$$\frac{a(\Lambda)}{a(\Delta)} = \prod_{p^r \| l, p^m \| n} \lambda_p(r, m; u_0) p^{r(k-2)+m(k-3/2)}.$$

Similarly,

$$\left(\frac{\det \Lambda}{\det \Delta}\right)^{-s} = \prod_{p^r \| l, p^m \| n} p^{-2s(r+m)}.$$

Additionally, from Lemmas 2.1 and 2.2 and a simple induction argument, it is easy to see that

$$\frac{\mu(\Lambda)}{\mu(\Delta)} = \prod_p \frac{\mu(\Lambda_r(u_0))}{\mu(\Lambda_0(u_0))} = \prod_{p^r \| l, p^m \| n} \mu_p(p^r; v_0)$$

Consequently, the inner sum in (3.1) equals

$$\sum_{l,n \geq 1} \prod_{p^r \| l, p^m \| n} \lambda_p(r, m; u_0) \mu_p(p^r; v_0) p^{r(k-2-2s)+m(k-3/2-2s)}.$$

This is easily seen to equal the expression in (3.2).

To complete the proof, we have to relate the expression in (3.2) to the generating functions in Propositions 2.2 and 2.3. The double sum on $r, m \geq 0$ is

$$\left(1 + \frac{v_0}{p}\right) \frac{N_p(x, y; u_0)}{G_p(x) H_p(y)} - \left(\frac{v_0}{p}\right) \frac{N_p(0, y; u_0)}{H_p(y)},$$

with $x = p^{-2s+k-1}$ and $y = p^{-2s+k-3/2}$ when $p \nmid N$ and $x = p^{-2s-1/2}\lambda$ and $y = p^{-2s}\lambda$ when $p|N$. This yields the result in the statement of the theorem. Note that only the numerator of the above expression depends on the local maximal type of Δ and that the denominator involving G_p and H_p is independent of Δ. □

Remark 3.1 *The function $Z_F(s)$ is the spinor zeta-function (cf. Andrianov [1]). The function $Z_F^{\times}(s-k+1)$ is the standard zeta-function (cf. Andrianov [2]). This should be compared to the result of Andrianov [1, p. 85] which, though more general in many respects, has only the spinor-zeta-function as a factor. Now $K_F(s)$, $Z_F(s)$ and $Z_F^{\times}(s)$ all have functional* *equations (see Imai [8], Andrianov [1], and Böcherer [5] or Andrianov-Kalinin [3], respectively). The interesting (and curious) observation is that as combined in the statement of the theorem, the functional equations are inconsistent ($s \mapsto k-s, k-1-s, k-1/2-s$, respectively). Breulmann-Kohnen [4] used the cited Andrianov result and a similar observation (with only two functional equations) to prove a multiplicity one theorem.*

Remark 3.2 *For a given maximal lattice Δ, the function $N_\Delta(s)$ is always an infinite Euler product. For a finite set of primes (those which divide the determinant of Δ, i.e., those where Δ is split), the local factor N_p will be of type Case 3. For asymptotically half the odd primes the local factor will be of type Case 1 and the other half of type Case 2. (For that finite set of primes that divide the level, the factors are a bit different but still involve the same three cases.) So, the "coefficient" of $a(\Delta)\mu(\Delta)(\det \Delta)^{-s}$ in the zeta-function is an Euler product that has structure analogous to a Dirichlet L-function. In a sense, we have expressed the Koecher-Maaß series in terms of a zeta-function whose coefficients are also zeta-functions.*

Remark 3.3 *The local factor $N_p(x;\Delta)$ in the above expression is a degree 6 polynomial in $x = p^{-2s}$, when $p \nmid N$ and $u_0 = 2, -1$, is degree 5 when $p \nmid N$ and $u_0 = 1$ and is quadratic if $p|N$.*

4 A strong inequality

Our final goal is to provide a strong bound for the average lattice-Fourier coefficients assuming the eigenvalues satisfy the Ramanujan-Petersson conjecture and the Genericity Condition. In the next proposition, we explicitly divide $\rho(r,m)$ by the factor $D(a_1,a_2)$ and $\rho(r)$ by $D(b_1)$. Recall that $\rho(r,m)$ is defined in (2.20) and $\rho(r)$ in (2.22).

Proposition 4.1 *The term $\rho(r,m)$ is "divisible" by $D(a_1,a_2)$ for all r,m. More explicitly,*

$$\begin{aligned}\frac{\rho(r,m)}{D(a_1,a_2)} = (a_1a_2)^{-m-r} \sum_{j=0}^{r} \sum_{i=0}^{[m/2]+j} & \epsilon(i+j)(a_1a_2)^{2i+\delta(i+j)} a_1^{m+2(j-i)} \\ & \times R(m+2(j-i), a_2/a_1) \\ & \times R(|m+2(r-i-j)+1|-1, (a_1a_2)^{\delta(i+j)}),\end{aligned}$$

where $R(v,x) = \sum_{l=0}^{v} x^l$, and

$$\delta(i) = \begin{cases} -1 & \text{if } i \geq m/2+r+1 \\ 0 & \text{else} \end{cases} \qquad \text{and} \quad \epsilon(i) = \begin{cases} -1 & \text{if } i \geq m/2+r+1 \\ 1 & \text{else} \end{cases}$$

Similarly, $D(b_1)$ divides $\rho(r)$ in the form $\sum_{l=0}^{r} b_1^{2l-r} = b_1^{-r} R(r, b_1^2)$.

Proof. The proof of the last statement is trivial so we skip that and give the details of the proof for $\rho(r, m)$.

By a simple rearrangement and collection of terms, we easily see that

$$\begin{aligned}
&(a_1 a_2)^{m+r+2} \rho(r, m) \\
&\quad = a_2^{m+1}(1 - a_1^{2m+2r+4})(1 - a_2^{2r+2}) \\
&\qquad - a_1^{m+1}(1 - a_1^{2r+2})(1 - a_2^{2m+2r+4}) \\
&\quad = (1 - a_1^2)(1 - a_2^2) a_2^{m+1} \sum_{j=0}^{r} a_2^{2j} \sum_{i=0}^{m+r+1} a_1^{2i} \\
&\qquad - (1 - a_1^2)(1 - a_2^2) a_1^{m+1} \sum_{j=0}^{r} a_1^{2j} \sum_{i=0}^{m+r+1} a_2^{2i} \\
&\quad = (1 - a_1^2)(1 - a_2^2) \sum_{j=0}^{r} \sum_{i=0}^{m+r+1} (a_1^{2i} a_2^{m+2j+1} - a_1^{m+2j+1} a_2^{2i}) \\
&\quad = (1 - a_1^2)(1 - a_2^2) \sum_{j=0}^{r} \sum_{i=0}^{m+r+1} \gamma(2i, m + 2j + 1),
\end{aligned}$$

where $\gamma(u, v) = a_1^u a_2^v - a_1^v a_2^u$. So, it suffices to show that the last double sum is divisible by $(a_2 - a_1)(1 - a_1 a_2)$ and find the quotient. Note that $\gamma(u, v)$ is already divisible by $(a_2 - a_1)$, but we ignore that fact for the moment.

Let $t = [m/2]$ and $S(r, m)$ be the double sum in the expression above. Write the inner sum on i (for fixed j) in two pieces. The first piece is for $0 \le i \le t + j$ and the second is for $t + j + 1 \le i \le m + r + 1$. So

$$S(r, m) = \sum_{j=0}^{r} \sum_{i=0}^{t+j} \gamma(2i, m + 2j + 1) + \sum_{j=0}^{r} \sum_{i=t+j+1}^{m+r+1} \gamma(2i, m + 2j + 1).$$

In the last double sum, first replace j by $r - j$ (invert the order of summation) and then replace i by $m+r+1-i$ (again invert). The resulting sum is on $0 \le j \le r$ and $0 \le i \le m - t + j$, and the summand is $\gamma(2m + 2r - 2i + 2, m + 2r - 2j + 1)$. If m is even, then $m - t = t$ so the upper limit on i is $t + j$ (as in the first sum above). If m is odd, then $m = 2t + 1$, so $m - t + j = t + j + 1$. For $i = t + j + 1$, the summand is zero, so we can ignore this term. Consequently, we can combine the two double sums to get

$$S(r,m) = \sum_{j=0}^{r}\sum_{i=0}^{t+j}[\gamma(2i, m+2j+1) + \gamma(2m+2r-2i+2, m+2r-2j+1)].$$

Now observe that $\gamma(v,u) = -\gamma(u,v)$ and $\gamma(u+w, v+w) = (a_1a_2)^w\gamma(u,v)$. Applying this to the second term within the sum above, we can write the summand in square-brackets as

$$(1-(a_1a_2)^{m+2(r-i-j)+1})\gamma(2i, m+2j+1)$$

or

$$(1-(a_1a_2)^{m+2(r-i-j)+1})(a_1a_2)^{2i}\gamma(0, m+2(j-i)+1).$$

Note that in the summation ranges, the second parameter in the last expression is always positive.

For $v+1 \geq 1$, $\gamma(0, v+1) = (a_2-a_1)a_1^vR(v, a_2/a_1)$, with R defined in the statement of the proposition. Similarly, if $v+1 \geq 1$, then

$$(1-x^{v+1}) = (1-x)R(v,x)$$

and if $v+1 \leq -1$, then

$$(1-x^{v+1}) = (1-1/x)R(|v+1|-1, 1/x) = -x^{-1}(1-x)R(|v+1|-1, x^{-1}).$$

and if $v+1=0$, this expression vanishes. To summarize this, we write

$$(1-x^{v+1}) = \epsilon_v x^{\delta_v}(1-x)R(|v+1|-1, x^{\delta_v}),$$

where $\epsilon_v = -1, \delta_v = -1$ if $v+1 \leq -1$ and $\epsilon_v = 1, \delta_v = 0$, otherwise.

With this notation, we then can write

$$\begin{aligned} S(r,m) \,=\, & (a_2-a_1)(1-a_1a_2)\sum_{j=0}^{r}\sum_{i=0}^{t+j}\epsilon(i+j)(a_1a_2)^{2i+\delta(i+j)}a_1^{m+2(j-i)} \\ & \times R(m+2(j-i), a_2/a_1) \\ & \times R(|m+2(r-i-j)+1|-1, (a_1a_2)^{\delta(i+j)}). \end{aligned}$$

where $\delta(i) = \delta_{m+2r-2i}$ and $\epsilon(i) = \epsilon_{m+2r-2i}$, as defined in the statement of the proposition. This completes the proof. □

We can use this proposition to give explicit bounds for the average lattice-Fourier coefficient (actually the local factor in that coefficient) whenever the Ramanujan-Petersson conjecture holds (see Weissauer [16]).

Theorem 4.1 *Let F be an eigenform that satifies both the Genericity Condition and the Ramanujan-Petersson conjecture. Let Δ be a maximal lattice containing a lattice Λ. Let p be a prime and write $\Lambda_{(p)} \simeq \Lambda_{r_p}(u_0)^{p^{m_p}}$. Write $l = \prod_p p^{r_p}$ and $n = \prod_p p^{m_p}$ and assume that $\gcd(ln, N) = 1$. Then for any $\epsilon > 0$,*

$$|a(\Lambda)| \ll_\epsilon |a(\Delta)|\, l^{k-2+\epsilon} n^{k-3/2+\epsilon}. \tag{4.1}$$

The implied constant depends only on ϵ and is computable.

Remark 4.1 *A weaker form of this result can be stated in the form*

$$|a(\Lambda)| \ll_{\Delta,\epsilon} l^{k-2+\epsilon} n^{k-3/2+\epsilon}.$$

Compare this to the conjectural bound of Resnikoff-Saldaña [12] (see also Böcherer-Raghavan [6], Raghavan [11] and Kohnen [9])

$$|a(\Lambda)| \ll_\epsilon (\det(\Lambda))^{k/2-3/4+\epsilon} \ll_{\Delta,\epsilon} (ln)^{k-3/2+\epsilon},$$

since $\det(\Lambda) = \det(\Delta)(ln)^2$. Thus, our theorem is in one sense stronger than the conjectured bounds in l-dependence, but weaker in that the result is relative to Δ. Note, however, that $\det(\Delta)$ is always square-free and may or may not have any prime factors in common with l and n.

This theorem then indicates two things. First, the optimal bound for Fourier coefficients is not *a function of the determinant itself, but depends naturally (and more precisely) on the local structure of the lattices. Second, the independence of Hecke eigenvalues and Fourier coefficients for maximal lattices creates a fundamental barrier to methods which attempt to bound Fourier coefficients (in absolute terms) based on bounds for eigenvalues.*

Remark 4.2 *The dependence on ϵ in the inequality could be replaced by some powers of* log *and other explicit (well-known) arithmetic functions, if necessary. We state the results in the above form only for simplicty. Additionally, the* gcd *assumption is also only for simplicity, in order to avoid discussion of the cases where $p|N$.*

Remark 4.3 *A slightly stronger form of the inequality (without implied constants) can be given in the case that $\Lambda = \Delta^n$, that is, if $l = 1$.*

Proof. From Proposition 4.1 we can easily derive a bound of the form

$$\left|\frac{\rho(r,m)}{D(a_1,a_2)}\right| \le \sum_{j=0}^{r} \sum_{i=0}^{[m/2]+j} (m+2(j-i)+1)|m+2(r-i-j)+1| \ll (m+r)^4,$$

with a computable implied constant. From Proposition 2.4 and (2.21) we see that

$$|\lambda_p(r,m;u_0)| \ll (1-1/\sqrt{p})^{-4}(m+r)^4,$$

with the same implied constant. This together with Theorem 2.1 implies the result. □

5 Open questions and future work

We hope to investigate some of the following open questions in the future.

1. What can we say about other zeta-functions (e.g., Rankin-type) in this context? For example, do our results provide a second proof of Theorem 5.1.1 (for $n = 2$) in Andrianov [2]?

2. Is the "lattice-exponential" defined in (1.1) *naturally* a Whittaker function of some type?

3. Is there a more fundamental role for $\rho(r, m)$ other than as a building block for the average lattice-Fourier coefficients?

4. Is there a two-complex variable zeta-function, analytic and with functional equation in each variable, for which the Koecher-Maaß series is a one-variable specialization and which explains the apparent inconsistencies of the functional equations for the three zeta functions in Theorem 3.1?

5. Perhaps the most interesting question is: how far can these ideas be pushed in degree greater than two?

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