COUNTABILITY

We say that a set X is **countable** if there is a bijective function $f : \mathbb{N} \to X$.

With such a set X and function f, we can set

$$x_1 = f(1),\ x_2 = f(2),\ x_3 = f(3),\ \ldots$$

and in this way enumerate the elements of X.

Recall that $\mathbb{N} \times \mathbb{N}$, or equivalently, $\mathbb{Z}_+ \times \mathbb{Z}_+$, denotes the set of all pairs of positive integers.

We claim that $\mathbb{N} \times \mathbb{N}$ (an infinite set) is countable, meaning that we have a one-to-one correspondence between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$.

A grid of the ordered pairs of positive integers

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Counting: ONE

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Counting: TWO

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Counting: THREE

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Counting: FOUR

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Counting: FIVE

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & {\color{red}(2,2)} & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Counting: SIX

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Counting: SEVEN

$$
\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

Counting: EIGHT

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Counting: NINE

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & {\color{red}(3,2)} & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Counting: TEN

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
\color{red}{(4,1)} & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Counting: ELEVEN

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Counting: TWELVE

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Counting: THIRTEEN

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & \color{red}{(3,3)} & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Counting: FOURTEEN

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Counting: FIFTEEN

$$\begin{array}{cccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Formula? Note that we have been counting along successive cross-diagonals.

$$\begin{array}{cccccc} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & \cdots \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & \cdots \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & \cdots \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & \cdots \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{array}$$

$(3, 2)$ is the 3rd pair on the 4th cross-diagonal.

On the first 3 cross-diagonals, there are $1 + 2 + 3 = 6$ pairs.

So the pair $(3, 2)$ is the $3 + 6$th pair that we count.

Note that the pairs on the kth cross-diagonal are

$$(1, k),\ (2, k - 1),\ (3, k - 2),\ \ldots,\ (k, 1).$$

More generally:
The pair (m, n) is the mth pair on the $(m + n - 1)$st cross-diagonal.

There are $m + n - 2$ cross-diagonals that precede this, and together they have

$$1 + 2 + 3 + \cdots + (m + n - 2) = \frac{(m + n - 2)(m + n - 1)}{2}$$

pairs.

Thus (m, n) is the $\frac{(m+n-2)(m+n-1)}{2} + m$th pair that we count.

Recall that a function is bijective if and only if it has an inverse.

We define $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by

$$g(m,n) = \frac{(m+n-2)(m+n-1)}{2} + m.$$

Then the inverse function $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ is given as follows.
For $k \in \mathbb{N}$, let ℓ be the largest integer so that

$$\frac{(\ell-2)(\ell-1)}{2} < k.$$

Then we let $m = k - \frac{(\ell-2)(\ell-1)}{2}$, and we let $n = \ell - m$.

We define $f(k) = (m,n)$.

One checks that f is the inverse of g, and hence f is bijective.

We can use this to show that there are **at least** as many positive integers as there are positive rational numbers (which are all numbers of the form $\frac{m}{n}$ where m and n are positive integers).

We define $h : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}_+$ by $h(m, n) = \frac{m}{n}$.
So each element of $\mathbb{Q}_+$ corresponds to at least one element of $\mathbb{N} \times \mathbb{N}$.

In fact, since for every $m, n \in \mathbb{N}$, we have

$$\frac{m}{n} = \frac{2m}{2n} = \frac{3m}{3n} = \frac{4m}{4n} = \cdots ,$$

each element of $\mathbb{Q}_+$ corresponds to infinitely many elements of $\mathbb{N} \times \mathbb{N}$.

With $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ the bijective function as above, we get $h \circ f : \mathbb{N} \to \mathbb{Q}_+$ and this function is surjective.

Infinity is WEIRD!