Analysis Workshop
Distribution of Rational and Irrational Numbers

Story. In this workshop, you prove something you may have been told for many years: between any two rational numbers is an irrational number, and between any two irrational numbers is a rational number. We have already seen the story that early Pythagoreans at first rejected the existence of irrational numbers, but were forced to accept the existence of $\sqrt{2}$ as a number occurring in nature, and a number they could prove is not rational. Subsequently, Eudoxus (c. 408-355 BC) is credited with developing theory regarding irrationals (see book V of Euclid's *Elements*). While the Pythagoreans' (arithmetical) theory was based on commensurable and incommensurable magnitudes, Eudoxus took a purely geometric approach, obviating the need to refer to commensurable and incommensurable magnitudes. Dedekind (1831-1916) took an approach quite similar to that of Eudoxus, and in 1872, published a construction of real numbers using what we now call "Dedekind cuts"; see, for instance

https://en.wikipedia.org/wiki/Dedekind_cut

Also in 1872, Cantor (1845-1918) published a construction of the irrational numbers, wherein each irrational number is the "limit" of a sequence of rational numbers (and to reflect this phenomenon, we say that the rational numbers are "dense" in the set of real numbers). Further, Cantor introduced a theory of "transfinite cardinal numbers" (roughly speaking, these are "numbers" that reflect different "magnitudes" of infinities). However, even though between any two rational numbers is an irrational number, and between any two irrational numbers is a rational number, the "magnitude" of the set of irrational numbers is larger than the "magnitude" of the set of rational numbers (this is a result due to Cantor, whose proof you will see in Introduction to Proofs and Group Theory).

For further exploration:

(a) Kronecker (1823-1891, mainly known for his contributions to number theory and to algebra) strongly objected to Cantor's theories, considering Cantor a scientific charlatan and a "corrupter of youth". One can read about this in the book *Great Feuds in Mathematics : Ten of the Liveliest Disputes Ever* by H. Hellman. (Note that Kronecker also rejected Weierstrass' construction of a continuous, nowhere differentiable function.)

(b) Another potentially interesting book is *To Infinity and Beyond: A Cultural History of the Infinite* by E. Maor.